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SCIENCE DEPT.

Geometry (I.C.)

This was translated by Carlyle

ELEMENTS OF GEOMETRY

AND

TRIGONOMETRY;

WITH NOTES.

TRANSLATED FROM THE FRENCH OF

M. A. LEGENDRE,

MEMBER OF THE INSTITUTE, AND OF THE LEGION OF HONOUR, AND
OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH, &c.

EDITED BY

DAVID BREWSTER, LL. D.

FELLOW OF THE ROYAL SOCIETY OF LONDON, AND SECRETARY
TO THE ROYAL SOCIETY OF EDINBURGH, &c. &c.

WITH NOTES AND ADDITIONS, AND AN INTRODUCTORY
CHAPTER ON PROPORTION.

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M. Le Chevalier Legendre
with the Best Respects
of the Editor

TO

WILLIAM WALLACE, Esq. F. R. S. E. &c.

AND

Profeſſor of Mathematics

IN THE

UNIVERSITY OF EDINBURGH,

THIS TRANSLATION OF

LEGENDRE'S ELEMENTS OF GEOMETRY

IS INSCRIBED,

IN TESTIMONY OF THE RESPECT AND ESTEEM

OF

THE EDITOR.

PREFACE OF THE EDITOR.

THE "Elements of Geometry and Trigonometry," by M. Le Chevalier Legendre, are so well known, and have been so highly esteemed in every part of Europe, that no apology can be required for presenting a translation of them to the English reader. The original work has undergone no fewer than *eleven* large impressions, and its distinguished author is at present occupied in preparing the *twelfth* edition for the press.

Upon announcing to M. Legendre my intention of superintending an English translation of his Elements of Geometry, and requesting his sanction and assistance, he kindly transmitted to me the following materials, with which the present edition has been enriched.

I. A new demonstration of the solidity of the pyramid, discovered by M. Queret of St Malo, which is substituted in place of the former demonstration, and forms Prop. XVII. and XVIII. of Book VI.*

M. Queret has been anticipated in this demonstration by the late Professor Playfair.

II. A reply to the objections of Mr Leslie against M. Legendre's theory of parallel lines.

III. A defence of the same theory by M. Le Baron Maurice.

Besides these additions communicated by the author, an Introduction on Proportion, by the translator, has been prefixed to the work, and several valuable trigonometrical tables have been added from the *Trigonométrie of Cagnoli.*

The greatest improvement, however, which has been made in this edition consists in the substitution of wooden cuts in the text, in place of the engraved plates which accompany all the editions of the original work. This improvement, which will be found of great use both to the teacher and the student, was considered indispensable in an English work.

D. B.

EDINBURGH, *August* 1, 1822.

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INTRODUCTION.

ON PROPORTION.

The doctrine of proportion belongs properly to Arithmetic, and ought to be explained in works which treat of that science. Its object being to point out the relations which subsist among magnitudes in general, when viewed as *measured*, or represented by *numbers*, the connexion it has with Geometry is not more immediate than with many other branches of knowledge, except indeed as Geometry affords the largest class of magnitudes capable of being so measured or represented, and thus offers the widest field for reducing it to practice. Owing, however, to our general and long-continued employment of Euclid's Elements, the fifth Book of which is devoted to proportion, our common systems of Arithmetic, and even of Algebra, pass over the subject in silence, or allude to it so slightly as to afford no adequate information. For the sake of the British student, therefore, it will be requisite to prefix a brief outline of the fundamental truths connected with this department of Mathematics; at least in so far as a knowledge of them is essential for understanding the work which follows.

The proper mode of treating proportion has given rise to much controversy among mathematicians; chiefly originating from the difficulties which occur in the application of its theorems to that class of magnitudes denominated *incommensurable*, or having no *common measure*. Euclid evades this obstacle; but his method is cumbrous, and, to a learner, difficult of comprehension. All other methods have the disadvantage of frequently employing the principle of *reductio ad absurdum*, a species of reasoning, which, though perfectly conclusive, the mathematician wishes to employ as seldom as possible. The opposite advantages, however, have generally overcome this reluctance; and Euclid's method is now almost entirely abandoned in elementary treatises. On this matter, we are happily delivered from the necessity of making any selection; the author having himself provided for the application of proportion to incommensurable quantities, and demonstrated every case of this kind as it occurred, by means of the *reductio ad absurdum*. He has also in various parts of these Elements interspersed explanations of the sense in which geometrical mag-

For, by hypothesis, we have $\left\{\begin{array}{l} n\,\mathrm{A} = m\,\mathrm{B} \\ q\,\mathrm{B} = p\,\mathrm{A} \end{array}\right\}$; and multiplying[*] together the terms which stand above each other, we obtain

$$nq\,\mathrm{AB} = mp\,\mathrm{AB};\ \text{hence } nq = mp;\ \text{hence } nq\mathrm{C} = mp\mathrm{C}.$$

But by hypothesis we have

$$n\,\mathrm{C} = m\,\mathrm{D};\ \text{hence } nq\,\mathrm{C} = mq\,\mathrm{D}.$$

Now we have already shewn

$$nq\,\mathrm{C} = mp\,\mathrm{C};\ \text{hence } mp\,\mathrm{C} = mq\,\mathrm{D},\ \text{hence } p\,\mathrm{C} = q\,\mathrm{D}.$$

III. The *ratio* or relation which is perceived to exist between two magnitudes of the same kind, when considered as mere magnitudes, appears to be a simple idea, and therefore unsusceptible of any good definition. It may be illustrated by observing, that when we have A : B : : C : D, the ratio of A to B is said to be *the same* as that of C to D. In Arithmetic, the ratio of two numbers is usually represented by their quotient, or by the fraction which results from making the first of them numerator, and the second denominator. It is in this sense, that one ratio is said to be *equal to* or *less* or *greater than* another ratio.

IV. The first and third terms of a proportion are called the *antecedents*; the second and fourth the *consequents*. The first and fourth are likewise called the *extreme terms*, or the *extremes*; the second and third *mean terms* or *means*. When both the means are the same, either of them is called a *mean proportional* between the two extremes; and if in a series of proportional magnitudes each consequent is the same as the next antecedent, those magnitudes are said to be in *continued proportion*.

Thus if we have A : B : : C : D, A and C are antecedents, B and D are consequents; A and D are extremes, B and C are means. If we have A : B : : B : C : : C : D : : D : E, B is a mean proportional between A and C, C between B and D, D between C and E; and the magnitudes A, B, C, D, E are said to be in continued proportion, or sometimes, in geometrical progression.

---

[*] Unless one of the magnitudes A and B is a number, they evidently cannot, in a literal sense, be multiplied together. The expression *product* of A and B must therefore, in all such cases, be regarded as elliptical, or employed merely for the sake of brevity. What is understood by it, the Author has explained at *pp.* 48. 49.

THEOREM I.

If four magnitudes are proportional, the product of the extremes will be equal to that of the means; and conversely, if two products are equal, any two factors composing the first will form the extremes of a proportion, in which any two factors composing the second form the means.

First. Suppose $A : B :: C : D$; then is $AD = BC$.

Find a common measure of A and B, if they have one: suppose it to be E; and that $A = m\,E$, $B = n\,E$. Then we shall have $n\,A = m\,B$; and therefore (*Def.* 2.) $n\,C = m\,D$. Equal quantities multiplied by equal quantities yield equal products; hence

$$n\,A \times m\,D = m\,B \times n\,C, \text{ or}$$
$$n\,m\,AD = n\,m\,BC, \text{ or}$$
$$AD = BC.^{*}$$

Secondly. If we have $AD = BC$; then we are to prove that $A : B :: C : D$.

Find the common measure of A and B; and suppose we have $n\,A = m\,B$.

$$\left.\begin{array}{r} \text{Now we have } AD=BC, \\ \text{Also } \quad mB=nA \end{array}\right\} ;\ \text{hence by multiplying,}$$
$$m\,DAB = n\,CAB, \text{ hence by dividing}$$
$$m\,D = n\,C;\ \text{and therefore } (\textit{Def. } 2.)$$
$$A : B :: C : D.$$

---

[*] If A and B are incommensurable, still we shall have $AD=BC$. For, if not, one of them must be greater: suppose AD greater, and that we have $AD=BC+CF$. By the method explained in *Def.* 2., find a measure of A which shall be less than F; and suppose $n\,A=m\,(B-B')$, B' being of course less than the measure, and therefore still less than F. By the same Definition, we shall have $n\,C=m\,(D-D')$, D' being a positive quantity. Hence we have

$$n\,A\times m(D-D')=n\,C\times m\,(B-B');\ \text{that is}$$
$$n\,m\,AD-n\,m\,AD'=n\,m\,CB-n\,m\,CB';\ \text{or dividing,}$$
$$AD - AD' = CB-CB';\ \text{but by the supposition, we had}$$
$$AD = CB-CF.$$

Hence $AD-AD$ being less than AD, $CB-CB$ must also be less than $CB-CF$; hence CB' is greater than CF, and B' is greater than F. On the contrary, however, it is less than F by hypothesis: hence that hypothesis was false; hence AD is not less than BC. We could shew in the same manner that it is not greater: hence it is equal to BC.

For, the first analogy gives

$$AD = BC,$$

the second gives $\underline{EH = FG;}$

therefore $AD : EH = BC : FG$ by multiplication,

or $AE \,.\, DH = BF \,.\, CG.$

Hence, Theor. I., we have

$$AE : BF :: CG : DH.$$

And the same reasoning would extend to any number of analogies.

Cor. 1. If the second analogy were the same as the first, we should have $A^2 : B^2 :: C^2 : D^2$; hence *the squares of proportional numbers are proportional.* The same is evidently true of the cubes or any other powers.

Cor. 2. Suppose we have the continued proportion $A : B :: B : C :: C : D$; then,

First. Having $A : B :: B : C,$

and $\underline{A : B :: A : B,}$

we shall have $A^2 : B^2 :: BA : BC$;

or(Cor. 2. Theor. 11.) $A^2 : B^2 :: A : C.$

Hence *in a continued proportion, the first is to the third as the square of the first is to the square of the second.* The ratio which A bears to C is sometimes called the *duplicate* of that which it bears to B.

Secondly. Having $A : B :: B : C,$

$A : B :: C : D,$

and $\underline{A : B :: A : B,}$

we shall have $A^3 : B^3 :: BCA : BCD$;

or (Cor. 2. Theor. I.) $A^3 : B^3 :: A : D.$

Hence *in continued proportionals, the first is to the fourth, as the cube of the first is to the cube of the second.* The ratio $A^3 : B^3$, or $A : D$, is sometimes called the *triplicate* of $A : B$; $A^4 : B^4$ the *quadruplicate;* and so on. The law which continued proportionals observe in regard to such ratios is now apparent.

By means of these Theorems, and their Corollaries, it is easy to demonstrate, or even to discover, all the most important facts connected with the doctrine of proportion. The facts given here will enable the student to go through these Elements, without any obstruction on that head.

ELEMENTS OF GEOMETRY.

BOOK I.

THE PRINCIPLES.

Definitions.

I. GEOMETRY is the science which has for its object the measurement of space.

Space has three dimensions, length, breadth, and height.

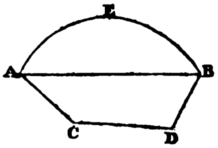
II. A *line* is length without breadth.

The extremities of a line are called *points:* a point, therefore, occupies no space.

III. A *straight line* is the shortest distance from one point to another.

IV. Every line, which is not straight, or composed of straight lines, is a *curve line.*

Thus, AB is a straight line; ACDB is a *broken* line, or one composed of straight lines; and AEB is a curve line.



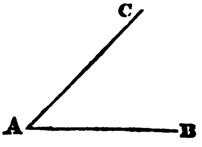
V. A *surface* is that which has length and breadth, without height or thickness.

VI. A *plane* is a surface, in which, if two points be assumed at will, and connected by a straight line, that line will lie wholly in the surface.

VII. Every surface, which is not plane, or composed of plane surfaces, is a *curve surface.*

VIII. A *solid* or *body* is that which combines all the three dimensions of space.

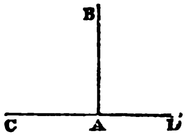
IX. When two straight lines, AB, AC, meet together, the quantity, greater or less, by which they are separated from each other in regard to their position, is called an *angle;* the point of intersection A is the *vertex* of the angle; the lines AB, AC, are its sides.



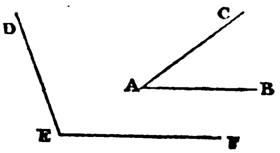
The angle is sometimes designated simply by the letter at the vertex A; sometimes by three letters BAC or CAB, the letter at the vertex being always placed in the middle.

Angles, like all other quantities, are susceptible of addition, subtraction, multiplication, and division. Thus the angle DCE (see Fig. to Prop. 4.) is the sum of the two angles, DCB, BCE; and the angle DCB is the difference of the two angles DCE, BCE.

X. When a straight line AB meets another straight line CD, so as to make the adjacent angles BAC, BAD equal to each other, each of those angles is called a *right angle;* and the line AB is said to be *perpendicular* to CD.



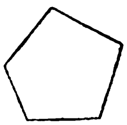
XI. Every angle BAC, less than a right angle, is an *acute angle;* every angle DEF, greater than a right angle, is an *obtuse angle.*



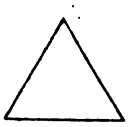
XII. Two lines are said to be *parallel*, when, being situated in the same plane, they cannot meet, how far soever both of them be produced.

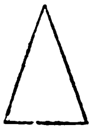
XIII. A *plane figure* is a plane terminated on all sides by lines.

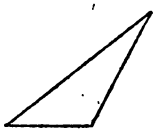
If the lines are straight, the space they enclose is called a *rectitioneal figure* or *polygon*, and the lines themselves taken together form the contour or *perimeter* of the polygon.



XIV. The polygon of three sides, the simplest of all, is called a *triangle;* that of four sides, a *quadilateral;* that of five, a *pentagon;* that of six, a *hexagon;* and so on.

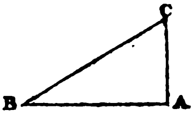






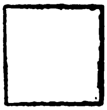
XV. An *equilateral* triangle is one which has its three sides equal; an *isosceles* triangle, one which has two of its sides equal; a *scalene* triangle, one which has its three sides unequal.

XVI. A right-angled triangle is one which has a right angle. The side opposite the right angle is called the *hypotenuse.* Thus, ABC is a triangle right-angled at A; the side BC is its hypotenuse.



XVII. Among quadrilaterals, we distinguish:

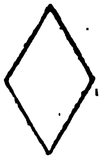
The *square*, which has its sides equal, and its angles right. (See Prop. 28. I.)



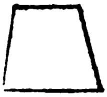
The *rectangle*, which has its angles right, without having its sides equal. (See the same Prop.)

The *parallelogram*, or *rhomboid*, which has its opposite sides parallel.

The *lozenge*, or *rhombus*, which has its sides equal, without having its angles right.



And, lastly, the *trapezium*, only two of whose sides are parallel.



XVIII. A *diagonal* is a line which joins the vertices of two angles not adjacent to each other. AC, in the diagram of Prop. 28, is a diagonal.

XIX. An *equilateral* polygon is one which has all its sides equal; an *equiangular* polygon, one which has all its angles equal.

XX. Two polygons are *mutually equilateral*, when they have their sides equal each to each, and placed in the same order; that is to say, when following their perimeters in the same direction, the first side of the one is equal to the first side of the other, the second of the one to the second of the other, the third to the third, and so on. The phrase, *mutually equiangular*, has a corresponding signification.

In both cases, the equal sides, or the equal angles, are named *homologous* sides or angles.

Note. In the first four books we shall treat exclusively of plane figures, that is, of figures traced on a plane surface.

Explanation of Terms and Signs.

An *axiom* is a self-evident proposition.

A *theorem* is a truth, which becomes evident by means of a train of reasoning called a *demonstration.*

A *problem* is a question proposed, which requires a *solution.*

A *lemma* is a subsidiary truth, employed for the demonstration of a theorem, or the solution of a problem.

The common name, *proposition,* is applied indifferently to theorems, problems, and lemmas.

A *corollary* is an obvious consequence deduced from one or several propositions.

A *scholium* is a remark on one or several preceding propositions, which tends to point out their connexion, their use, their restriction, or their extension.

An *hypothesis* is a supposition, made either in the enunciation of a proposition, or in the course of a demonstration.

The sign $=$ is the sign of equality; thus, the expression $A=B$, signifies that A is equal to B.

To signify that A is smaller than B, the expression $A<B$ is used.

To signify that A is greater than B, the expression $A>B$ is used.

The sign $+$ is pronounced *plus:* it indicates addition.

The sign $-$ is pronounced *minus:* it indicates subtraction.

Thus, $A+B$ represents the sum of the quantities A and B; $A-B$ represents their difference, or what remains after B is taken away from A; and $A-B+C$, or $A+C-B$, signifies, that A and C are to be added together, and that B is to be deducted from the whole.

The sign $\times$ indicates multiplication; thus $A\times B$ represents the product of A and B. Instead of the sign $\times$, a point is sometimes employed; thus, $A.B$ is the same thing as $A\times B$. The same product is also designated without any intermediate sign by AB; but this expression cannot be employed, when there is any danger of confounding it with that of the line AB, the distance between the points A and B.

The expression $A\times(B+C-D)$ represents the product of A by the quantity $B+C-D$. If $A+B$ were to be multiplied by $A-B+C$, the product would be indicated thus, $(A+B)\times(A-B+C)$, whatever is enclosed within a parenthesis being considered as a single quantity.

A number placed before a line, or a quantity, serves as a multiplier to that line or quantity; thus, 3 AB signifies that the line AB is taken three times; ½ A signifies the half of the angle A.

The square of the line AB is designated by AB^2; its cube by AB^3. What is meant by the square and the cube of a line will be explained in its proper place.

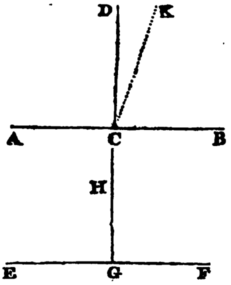
The sign $\surd$ indicates a root to be extracted; thus, $\surd 2$ means the square-root of 2; $\surd A \times B$ means the square-root of the product of A and B, or the mean proportional between them.

Axioms.

1. Two quantities equal to a third are equal to each other.
2. The whole is greater than its part.
3. The whole is equal to the sum of the parts into which it has been divided.
4. From one point to another, only one straight line can be drawn.
5. Two magnitudes, lines, surfaces, or solids, are equal, if, when applied to each other, they coincide throughout their whole extent.

PROPOSITION I. THEOREM.

All right angles are equal to each other.



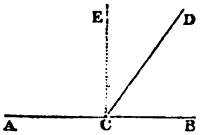
Let the straight line CD be perpendicular to AB, and GH to EF; the angles ACD and EGH will be equal to each other.

Take the four distances CA, CB, GE, GF all equal; the distance AB will be equal to the distance EF, and the line EF being placed on AB, so that the point E falls on A, the point F may be made to fall on B. Those two lines will thus coincide entirely; for otherwise there would be two straight lines extending from A to B, which (*Ax.* 4.) is impossible: and hence G, the middle point of EF, will fall on C, the middle point of AB. The side GE being thus applied to CA, the side GH must fall on CD. For suppose, if possible, that it falls on a line CK different from CD: then, since by hypothesis (*def.* 10.) the angle EGH=HGF, ACK would in that case be equal to KCB. But the angle ACK is greater than

ACD; and KCB is smaller than BCD, and by hypothesis ACD=BCD; hence ACK is greater than KCB. Therefore the line GH cannot fall on a line CK different from CD; therefore it falls on CD, and the angle EGH on ACD; therefore all right angles are equal to each other.

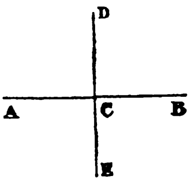
PROPOSITION II. THEOREM.

Every straight line CD, *which meets another* AB, *makes with it two adjacent angles* ACD, BCD, *the sum of which is equal to two right angles.*



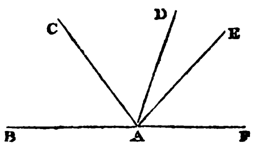
At the point C, erect CE perpendicular to AB. The angle ACD is the sum of the angles ACE, ECD; therefore ACD+BCD will be the sum of the three ACE, ECD, BCD: but the first of those three is a right angle, and the other two together make up the right angle BCE; hence the sum of the two angles ACD and BCD is equal to two right angles.

Cor. 1. If one of the angles ACD, BCD is right, the other must be right also.



Cor. 2. If the line DE is perpendicular to AB, in like manner AB will be perpendicular to DE.

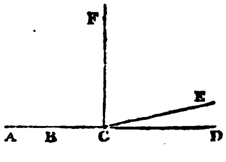
For, since DE is perpendicular to AB, the angle ACD must be equal to its adjacent one DCB, and both of them must be right. But since ACD is a right angle, its adjacent one ACE must also be right: hence the angle ACE=ACD; hence AB is perpendicular to DE.



Cor. 3. All the consecutive angles BAC, CAD, DAE, EAF formed on the same side of a straight line BF, are equal, when taken together, to two right angles; because their sum is equal to that of the two adjacent angles, BAC, CAF.

PROPOSITION III. THEOREM.

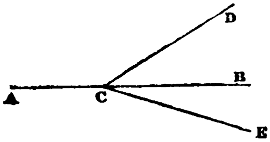
Two straight lines, which have two points common to both, coincide with each other throughout their whole extent, and form one and the same straight line.



Let A and B be the two common points. In the first place, it is evident that the two lines must coincide entirely between A and B, for otherwise there would be two straight lines between A and B, which is impossible (*Ax.* 4.). Suppose, however, that on being produced, those lines begin to separate at C, the one becoming CD, the other CE. From the point C draw the line CF, making with CA the right angle ACF. Now since ACD is a straight line, the angle FCD will be right (Prop. 2. cor. 1.); and since ACE is a straight line, the angle FCE will likewise be right. But the part FCE cannot be equal to the whole FCD; hence the straight lines which have two points A and B common, cannot separate in any point of their production; hence they form one and the same straight line.

PROPOSITION IV. THEOREM.

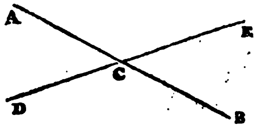
If two adjacent angles, ACD, DCB, *are together equal to two right angles, the two exterior sides,* AC, CB, *will lie in the same straight line.*



For if CB is not the production of AC, let CE be that production: then the line ACE being straight, the sum of the angles ACD, DCE, will (Prop. 2.) be equal to two right angles. But, by hypothesis, the sum of the angles ACD, DCB is also equal to two right angles: therefore, ACD+DCE must be equal to ACD+DCB; and taking away the angle ACD from each, there remains the part DCB equal to the whole DCE, which is impossible; therefore, CB is the production of AC.

PROPOSITION V. THEOREM.

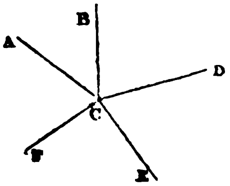
Whenever two straight lines AB, DE, *intersect each other, the opposite or vertical angles, which they form, are equal.*



For, since DE is a straight line, the sum of the angles ACD, ACE is equal to two right angles; and since AB is a straight line, the sum of the angles ACE, BCE is also equal to two right angles: hence the sum ACD+ACE is equal to the sum ACE+BCE. Take away from both the same angle ACE; there remains the angle ACD, equal to its opposite or vertical angle BCE.

It may be shewn, in the same manner, that the angle ACE is equal to its opposite angle BCD.

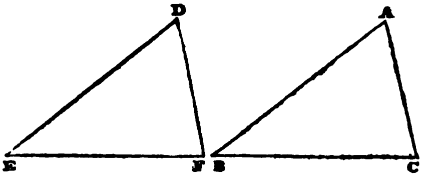
Scholium. The four angles formed about a point by two straight lines which intersect each other, are together equal to four right angles: for the two angles ACE, BCE, taken together, are equal to two right angles; and the other two, ACD, BCD have the same value.



In general, if any number of straight lines CA, CB, CD, &c. meet in a point C, the sum of all the consecutive angles ACB, BCD, DCE, ECF, FCA, will be equal to four right angles: for if four right angles were formed about the point C, by means of two lines perpendicular to each other, the same space would be occupied, either by the four right angles, or by the successive angles ACB, BCD, DCE, ECF, FCA.

PROPOSITION VI. THEOREM.

Two triangles are equal when an angle, and the two sides which contain it in the one, are respectively equal to an angle and the two sides which contain it in the other.



Let the angle A be equal to D, the side AC equal to the side DF, the side AB equal to DE; then will the triangle ABC be equal to DEF.

For those triangles may be applied to each other, so that they shall perfectly coincide. If the side DE be placed on its equal AB, the point D will fall on A, and the point E on B: and, since the angle D is equal to the angle A, when the side DE is placed on AB, the side DF will take the direction AC. Besides, DF is equal to AC; therefore, the point F will fall on C, and the third side EF will exactly cover the third side BC; therefore (*Ax.* 5.), the triangle DEF is equal to the triangle ABC.

Cor. When, in two triangles, these three things are equal, namely, the angle A=D, the side AB=DE, and the side AC=DF, the other three are equal also, namely, the angle B=E, the angle C=F, and the side BC=EF.

PROPOSITION VII. THEOREM.

Two triangles are equal, if two angles and the interjacent side of the one are equal to two angles and the interjacent side of the other.

Let the side BC (see the last figure) be equal to the side EF, the angle B to the angle E, and the angle C to the angle F; then will the triangle DEF be equal to the triangle ABC.

For, to bring about the superposition, let EF be placed on its equal BC; the point E will fall on B, and the point F on C. And, since the angle E is equal to the angle B, the side ED will take the direction BA; therefore, the point D will be found somewhere in the line BA. In like manner, since the angle F is equal to the angle C, the line FD will take the direction CA, and the point D will be found somewhere in the line CA. Hence, the point D, occurring at the same time in the two straight lines BA and CA, must fall on their intersection A; hence the two triangles ABC, DEF, coincide with each other, and are perfectly equal.

Cor. Whenever, in two triangles, these three things are equal, namely, $BC=EF$, $B=E$, $C=F$, it may be inferred that the other three are equal also, namely, $AB=DE$, $AC=DF$, $A=D$.

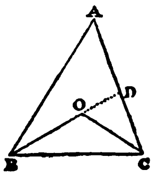
PROPOSITION VIII. THEOREM.

In every triangle, any side is less than the sum of the other two.

For the line BC, for example (see the preceding figure), is the shortest distance from B to C; therefore, BC is less than $BA+AC$.

PROPOSITION IX. THEOREM.

If, from a point O *assumed within the triangle* ABC, *straight lines* OB, OC, *be drawn to the extremities of a side* BC, *the sum of these straight lines will be less than that of the two other sides* AB, AC.

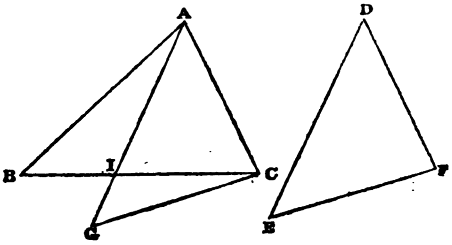


Let BO be produced till it meet the side AC in D. The line OC (Prop. 8.) is shorter than $OD+DC$: add BO to each, and we have $BO+OC \angle BO+OD+DC$, or $BO+OC \angle BD+DC$.

In like manner, $BD \angle BA+AD$: add DC to each; and we have $BD+DC \angle BA+AC$. But we have just found $BO+OC \angle BD+DC$; therefore, still more is $BO+OC \angle BA+AC$.

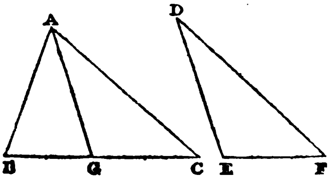
PROPOSITION X. THEOREM.

If the two sides AB, AC, *of the triangle* ABC, *are equal respectively to the two sides* DE, DF, *of the triangle* DEF, *while, at the same time, the angle* BAC *contained by the former, is greater than* EDF *contained by the latter; then will the third side,* BC *of the first triangle, be greater than the third* EF *of the second.*



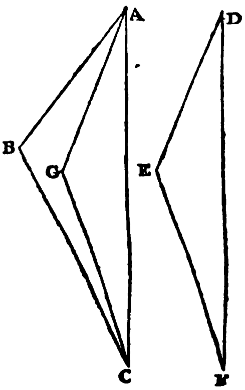
Make the angle CAG =D, take AG=DE, and join CG. The triangle GAC is equal to DEF (Prop. 6.), since, by construction, they have an equal angle in each, contained by equal sides; therefore CG is equal to EF. Now, there may be three cases in the proposition, according as the point G falls without the triangle ABC, or upon its base BC, or within it.

First Case. The straight line GC is shorter than GI+IC, and the straight line AB is shorter than AI+IB; therefore, GC+AB is shorter than GI+AI+IC+IB, or, which is the same thing, GC+AB $<$ AG+BC. Take away AB from the one side, and its equal AG from the other, there will remain GC $<$ BC; but we have found GC=EF, therefore, EF $<$ BC.



Second Case. If the point G fall on the side BC, it is evident that GC, or its equal EF, will be shorter than BC.

Third Case. Lastly, if the point G fall within the triangle ABC, we shall have, by the preceding theorem, $AG+GC \angle AB+BC$; and, taking AG from the one side, and its equal AB from the other, there will remain $GC \angle BC$, or $EF \angle BC$.



Scholium. Conversely, if the two sides AB, AC of the triangle ABC are equal to the two DE, EF of the triangle DEF, while the third side, CB of the first triangle, is greater than the third EF of the second; then will the angle BAC of the first triangle be greater than the angle EDF of the second.

For if not, the angle BAC must be equal to EDF, or less than it. In the first case, the side CB would (Prop. 6.) be equal to EF; in the second, CB would be less than EF: but both of these results contradict the hypothesis; therefore, BAC is greater than EDF.

PROPOSITION XI. THEOREM.

Two triangles are equal, when the three sides of the one are respectively equal to the three sides of the other.

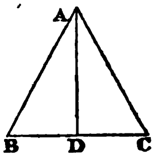
Let the side AB (see the diagram of Prop. 6.) be equal to DE, $AC=DF$, $BC=EF$; then is the angle $A=D$, $B=E$, $C=F$.

For, if the angle A were greater than D, since the sides AB, AC are respectively equal to DE, DF, it would follow, by the last proposition, that the side BC must be greater than EF: and, if the angle A were less than D, it would follow that the side BC must be less than EF. But BC is equal to EF; therefore the angle A can neither be greater nor less than D; therefore they are equal. In the same manner it may be shewn, that B is equal to E, and C to F.

Scholium. It may be observed, that the equal angles lie opposite to the equal sides: thus the equal angles A and D lie opposite to the equal sides BC and EF.

PROPOSITION XII. THEOREM.

In an isosceles triangle, the angles opposite the equal sides are equal.



Let the side AB be equal to AC, the angle C will be equal to B.

Join A the *vertex*, and D the middle point of the base BC. The triangles ADB, ADC, have all the sides of the one respectively equal to those of the other, AD being common, AB=AC *(hyp.)* and BD=DC by construction; therefore, by the last proposition, the angle B is equal to the angle C.

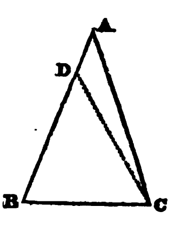
Cor. An equilateral triangle is likewise equiangular, that is to say, has all its angles equal.

Scholium. The equality of the triangles ABD, ACD, proves also that of the angle BAD to DAC, and of BDA to ADC; hence the latter two are right angles; *hence the line drawn from the vertex of an isoceles triangle to the middle point of its base, is perpendicular to that base, and divides the angle at the vertex into two equal parts.*

In a triangle which is not isosceles, any side may be assumed indifferently as the *base;* and the *vertex* is, in that case, the vertex of the opposite angle. In an isosceles triangle, however, that side is specially assumed as the base, which is not equal to either of the other two.

PROPOSITION XIII. THEOREM.

Conversely, if two angles of a triangle are equal, the sides opposite them will be equal, and the triangle will be isosceles.

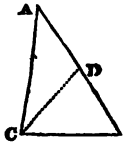


Let the angle ABC be equal to ACB; then will the side AC be equal to the side AB.

For, if those sides are not equal, let AB be the greater. Take BD=AC, and join DC. The angle DBC is *(Hyp.)* equal to ACB; the two sides DB, BC, are equal to the two AC, CB, by construction; therefore (Prop. 6.), the triangle DBC must be equal to ACB. But the part cannot be equal to the whole; hence there is no inequality between the sides AB, AC; hence the triangle ABC is isosceles.

PROPOSITION XIV. THEOREM.

Of two sides of a triangle, the greater is that which lies opposite the greater angle; and conversely, of two angles of a triangle, the greater is that which lies opposite the greater side.



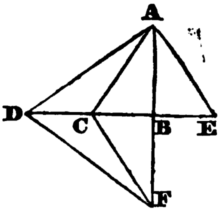
First, Let the angle C be greater than B; then will the side AB, opposite to C, be greater than AC, opposite to B.

Make the angle BCD=B. Then in the triangle BDC, we shall have BD=DC. (Prop. 13.) But the line AC is shorter than AD+DC, and AD+DC=AD+DB=AB; therefore, AB is greater than AC.

Secondly. Suppose the side AB $>$ AC; then will the angle C, opposite to AB, be greater than the angle B, opposite to AC. For, if we had C $<$ B, it would follow, from what has just been proved, that we must have AB $<$ AC, which is contrary to the hypothesis. If we had C=B, it would follow (Prop. 13.) that we must have AB=AC, which is also contrary to the hypothesis. Therefore, the angle C must be greater than B.

PROPOSITION XV. THEOREM.

From a given point A *without the straight line* DE, *no more than one perpendicular can be drawn to that line.*



For, suppose we can draw two, AB and AC. Produce one of them AB, till BF is equal to AB, and join FC.

The triangle CBF is equal ABC; for the angles CBF and CBA are right, the side CB is common, and the side BF=AB: therefore (Prop. 6.), those triangles are equal, and the angle BCF=BCA. The angle BCA is right, by hypothesis; therefore BCF must be right also. But if the adjacent angles BCA, BCF are together equal to two right angles, the line ACF must (Prop. 4.) be straight; from whence it follows, that between the same two points A and F, two straight lines can be drawn; which is impossible: hence it is equally impossible that two perpendiculars can be drawn from the same point, to the same straight line.

Scholium. At one given point C, in the line AB, it is equally impossible to erect two perpendiculars to that line; for (see the diagram of Prop. 2.), if CD and CE were those two perpendiculars, the angle DCB would be right, as well as BCE, and the part would thus be equal to the whole.

PROPOSITION XVI. THEOREM.

If from a point A (see the preceding figure), *situated without the straight line* DE, *a perpendicular* AB *be let fall on that straight line, and several oblique lines* AE, AC, AD, *be drawn to several points in the same line.*

First, *The perpendicular* AB *will be shorter than any oblique line.*

Secondly, *The two oblique lines,* AC, AE *drawn on different sides of the perpendicular, at equal distances* BC, BE, *will be equal.*

Thirdly, *Of two oblique lines,* AC, AD, *or* AE, AD, *drawn at pleasure, the one which lies farther from the perpendicular will be the longer.*

Produce the perpendicular AB till BF is equal to AB, and join FC, FD.

First. The triangle BCF, is equal to the triangle BCA, for they have the right angle CBF=CBA, the side CB common, and the side BF=BA; hence the third sides, CF and AC are equal. But ABF, being a straight line, is shorter than ACF, which is a broken line; therefore AB, the half of ABF, is shorter than AC, the half of ACF: therefore the perpendicular is shorter than any oblique line.

Secondly. If we suppose BE=BC; since we have, farther, the side AB common, and the angle ABE=ABC, the triangle ABE must be equal to the triangle ABC; hence the sides AE, AC are equal; hence two oblique lines equally distant from the perpendicular are equal.

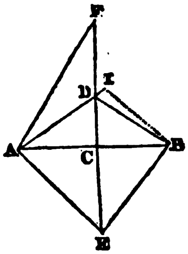
Thirdly. In the triangle DFA, the sum of the lines AC, CF, is less (Prop. 9.) than the sum of the sides AD, DF; therefore AC, the half of the line ACF, is shorter than AD, the half of the line ADF; therefore such oblique lines as lie farthest from the perpendicular are longest.

Cor. 1. The perpendicular measures the true distance of a point from a line, because it is shorter than any other distance.

Cor. 2. From the same point three equal straight lines cannot be drawn to the same straight line; for if there could, we should have two equal oblique lines on the same side of the perpendicular, which is impossible.

PROPOSITION XVII. THEOREM.

If from C, *the middle point of the straight line* AB, *the line* EF *be drawn perpendicular to this straight line, then* (first) *every point of the perpendicular will be equally distant from the two extremities of* AB; *and* (secondly) *every point situated without the perpendicular will be unequally distant from those extremities.*



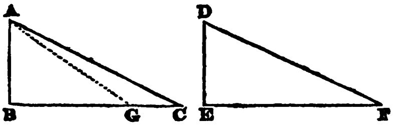
First. Since we suppose AC=CB, the two oblique lines AD, DB, are equally distant from the perpendicular, and therefore equal. So, likewise, are the two oblique lines AE, EB, the two AF, FB, and so on. Therefore every point in the perpendicular is equally distant from the extremities A and B.

Secondly. Let I be a point out of the perpendicular. If IA and IB be joined, one of those lines will cut the perpendicular in D; from which drawing DB, we shall have DB=DA. But the straight line IB is less than ID+DB, and ID+DB=ID+DA=IA; therefore IB<IA; therefore every point out of the perpendicular is unequally distant from the extremities A and B.

PROPOSITION XVIII. THEOREM.

Two right-angled triangles are equal, when the hypotenuse and a side of the one are respectively equal to the hypotenuse and a side of the other.

Suppose the hypotenuse AC=DF, and the side AB=DE; the right-angled triangle ABC will be equal to the right-angled triangle DEF.



Their equality would be manifest, if the third sides BC and EF were equal. If possible, suppose that those sides are not equal, and that BC is the greater. Take BG=EF; and join AG. The triangle ABG is equal to DEF; for the right angles B and E are equal, the side AB=DE, and BG=EF; hence these triangles are equal (Prop. 6.), and consequently AG=DF. Now *(Hyp.)* we have DF=AC; and therefore AG=AC. But (Prop. 16.) the oblique line AC cannot be equal to AG, which lies nearer the perpendicular AB; therefore it is impossible that BC can differ from EF; therefore the triangles ABC and DEF are equal.

PROPOSITION XIX. THEOREM.

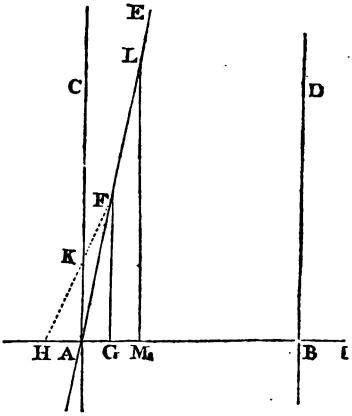
If two straight lines AC, BD, (see the next figure), *are perpendicular to a* third AB, *they will be parallel, in other words* (Def. 12.), *they will never meet, how far soever both of them be produced.*

For if they could meet in a point O, on either side of AB, there would be two perpendiculars OA, OB, let fall from the same point, on the same straight line, which is impossible.

PROPOSITION XX. LEMMA.

If the straight line BD *is perpendicular to* AB, *and if another straight line* AE *makes with* AB *the acute angle* BAE, *then the straight lines* BD, AE, *if produced sufficiently, will meet.*

From any point F, taken in the direction AE, let FG be drawn perpendicular to AB. The point G cannot fall on A, for the angle FAB is less than a right angle; it can still less fall on H in the production of BA, for then there would be two perpendiculars KA, KH, drawn from the same point K to the same straight line AH. Hence the point G must fall, as the figure represents, in the direction AB.



Now, let another point L be taken in the line AE, at a distance AL greater than AF, and let LM be drawn from it perpendicular to AB. We can prove, as in the preceding case, that the point M cannot fall on G, or in the direction GA; it must fall therefore in the direction OB; so that the distance AM will, of necessity, be greater than AG.

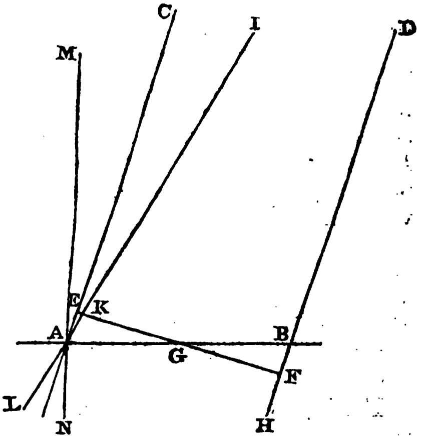
I observe farther, that if the figure be constructed with care, and AL be taken double of AF, we shall find that AM is exactly double of AG; in like manner, if AL is taken triple of

AF, we shall find that AM is triple of AG; and, in general, that there is always the same proportion between AM and AG, as between AL and AF. This proportion being settled, it follows not only that the straight line AE, if produced sufficiently, will meet BD, but also that the distance on AE, of this point of concourse, may be accurately assigned. It will be the fourth term of the proportion, AG : AB : : AF : x.

Scholium. The preceding investigation, being founded on a property which is not deduced from reasoning alone, but discovered by measurements made on a figure constructed accurately, has not the same character of rigorousness with the other demonstrations of elementary geometry. It is given here merely as a simple method of arriving at a conviction of the truth of the proposition. For a strictly rigorous demonstration we refer to the second Note.

PROPOSITION XXI. THEOREM.

If two straight lines, AC, BD, *form with a third* AB, *two interior angles* CAB, ABD, *whose sum is equal to two right angles, the lines* AC, BD, *will be parallel.*



From G, the middle point of AB, draw the straight line EGF perpendicular to AC. It will also be perpendicular to BD. For the sum GAE+GBD is (*Hyp.*) equal to two right angles; the sum GBF+GBD is (Prop. 2.) likewise equal to two right angles; and taking away GBD from both, there remains the angle GAE=GBF. Again, the angles AGE, BGF are equal (Prop. 5.), therefore the triangles AGE and BGF have each a side and two adjacent angles equal; therefore (Prop. 7.) they are themselves equal, and the angle BFG is equal to AEG: but AEG is a right angle by construction; therefore the line AC, BD, being perpendicular to the same straight line EF, are parallel (Prop. 19.).

PROPOSITION XXII. THEOREM.

If two straight lines AI, BD, *make with a third* AB, *two interior angles* BAI, ABD, *whose sum is less than two right angles, the lines* AI, BD *will meet if produced.*

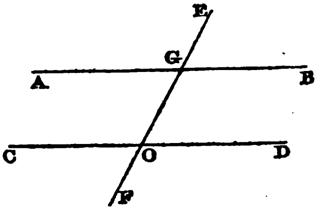
Draw the straight line AC (see the preceding figure), making the angle CAB=ABF, in other words, so that the angles CAB ABD, taken together, may be equal to two right angles; and complete the rest of the construction, as in the foregoing theorem. Since AEK is a right angle, AE the perpendicular is shorter than AK the oblique line; hence (Prop. 14.) in the triangle AEK, the angle AKE, opposite the side AE, is less than the right angle AEK, opposite the side AK. Hence the angle IKF, equal to AKE, is less than a right angle; hence (Prop. 20.) the lines KI and FD will meet if produced.

Scholium. If the lines AM, BD make with AB two angles BAM, ABD, the sum of which is greater than two right angles, those lines will not meet above AB, but they will below. For the two angles BAM, BAN are together equal to two right angles, and so are the two ABD, ABF; hence those four angles are together equal to four right angles. But BAM, ABD are together greater than two right angles; therefore the remaining angles BAN, ABF are together less than two; therefore the straight lines AN, BF will meet if produced.

Cor. Through a given point A, no more than one line can be drawn parallel to a given line BD. For there is but one line AC, which makes the sum of the two angles BAC+ABD equal to two right angles; and this is the parallel required. Every other line AI or AM would make the sum of the interior angles less or greater than two right angles; therefore it would meet BD.

PROPOSITION XXIII. THEOREM.

If two parallels AB, CD *are met by a secant* EF, *the sum of the interior angles* AGO, GOC *will be equal to two right angles.*



For if it were more or less, the two straight lines AB, CD would meet on the one side or the other (Prop. 22.), and would not be parallel.

Cor. 1. If GOC is a right angle, AGO will be a right angle also; therefore every line perpendicular to one of two parallels is perpendicular to the other.

Cor. 2. Since the sum $AGO+GOC$ is equal to two right angles, and the sum $GOD+GOC$ is also equal to two, if GOC be taken from both, there will remain the angle $AGO=GOD$. Besides (Prop. 5.), we have $AGO=BGE$, and $GOD=COF$; hence the four acute angles AGO, BGE, GOD, COF are equal to each other. The same is the case with the four obtuse angles AGE, BGO, GOC, DOF. It may be observed, moreover, that, in adding one of the acute angles to one of the obtuse, the sum will always be two right angles.

Scholium. The angles just spoken of, when compared with each other, assume different names. AGO, GOC, we have already named *interior angles on the same side;* BGO, GOD have the same designation; AGO, GOD are called *alternate interior* angles, or simply *alternate;* so also are BGO, GOC: and lastly EGB, GOD, or EGA, GOC are called, respectively, the opposite *exterior and interior* angles; and EGB, COF, or AGE, DOF the *alternate exterior* angles. This being premised, the following propositions may be considered as already demonstrated.

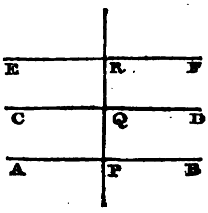
First. The interior angles on the same side are together equal to two right angles.

Second. The alternate interior angles are equal; so likewise are the opposite exterior and interior, and the alternate exterior angles.

Conversely, if in this second case, two angles of the same name are equal, the lines to which they refer will be parallel. Suppose, for example, the angle $AGO=GOD$. Since $GOC+GOD$ is equal to two right angles, $AGO+GOC$ must also be equal to two, and (Prop. 21.) the lines AG, CO must be parallel.

PROPOSITION XXIV. THEOREM.

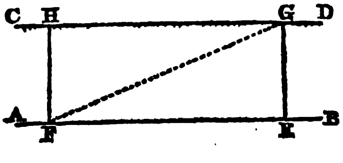
Two lines AB, CD, *parallel to a third, are parallel to each other.*



Draw PQR perpendicular to EF, and cutting AB, CD. Since AB is parallel to EF, PR will be perpendicular to AB (Prop. 23. Cor. 1.); and since CD is parallel to EF, PR will for a like reason be perpendicular to CD. Hence AB and CD are perpendicular to the same straight line; hence (Prop. 19.) they are parallel.

PROPOSITION XXV. THEOREM.

Two parallels are every where equally distant.

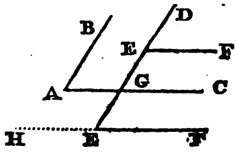


Two parallels AB, CD being given, if through two points assumed at pleasure, the straight lines EG, FH be drawn perpendicular to AB, those straight lines will (Prop. 23.) at the same time be perpendicular to CD: and we are now to shew that they will be equal to each other.

For, if GF be joined, the angles GFE, FGH, considered in reference to the parallels AB, CD, will be alternate interior angles, and therefore (Prop. 23. Schol.) equal to each other. Also, because the straight lines EG, FH are perpendicular to the same straight line AB, and consequently parallel, the angles EGF, GFH, considered in reference to the parallels EG, FH, will be alternate interior angles, and therefore equal. Hence the two triangles EFG, FGH have a common side, and two adjacent angles in each equal; hence these triangles (Prop. 7.) are equal; hence the side EG, which measures the distance of the parallels AB and CD at the point E, is equal to the side FH, which measures the distance of the same parallels at the point F.

PROPOSITION XXVI. THEOREM.

If two angles BAC, DEF *have their sides parallel, each to each, and lying in the same direction, those angles will be equal.*

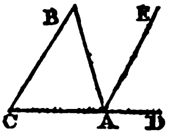


Produce DE, if necessary, till it meets AC in G. The angle DEF is equal to DGC (Prop. 23.), since EF is parallel to GC; and the angle DGC is equal to BAC, since DG is parallel to AB: hence the angle DEF is equal to BAC.

Scholium. The restriction of this proposition to the case where the side EF lies in the same direction with AC, is necessary, because if FE were produced towards H, the angle DEH would have its sides parallel to those of the angle BAC, but would not be equal to it. In that case, DEH and BAC would be together equal to two right angles.

PROPOSITION XXVII. THEOREM.

In every triangle, the sum of the three angles is equal to two right angles.



Let ABC be any triangle. Produce the side CA towards D; and, at the point A, draw AE parallel to BC.

Since AE, CB are parallel, and CAD cuts them, the exterior angle DAE will be equal to its interior opposite one ACB; in like manner, since AE, CB are parallel, and AB cuts them, the alternate interior angles ABC, BAE will be equal: hence the three angles of the triangle ABC make up the same sum as the three angles, CAB, BAE, EAD; hence (Prop. 2. Cor. 3.) that sum is equal to two right angles.

Cor. 1. Two angles of a triangle being given, or merely their sum, the third will be found by subtracting that sum from two right angles.

Cor. 2. If two angles of one triangle are respectively equal to two angles of another, the third angles will also be equal, and the two triangles will be mutually equiangular.

Cor. 3. In any triangle there can be but one right angle; for if there were two, the third angle must be nothing. Still less can a triangle have more than one obtuse angle.

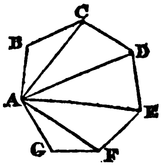
Cor. 4. In every right-angled triangle, the sum of the two acute angles is equal to one right angle.

Cor. 5. Since every equilateral triangle (Prop. 12.) is also equiangular, each of its angles will be equal to the third part of a right angle; so that if the right angle is expressed by unity, the angle of an equilateral triangle will be expressed by $\frac{2}{3}$.

Cor. 6. In every triangle ABC, the exterior angle BAD is equal to the sum of the two interior opposite angles B and C. For, AE being parallel to BC, the part BAE is equal to the angle B, and the other part DAE is equal to the angle C.

PROPOSITION XXVIII. THEOREM.

The sum of all the interior angles of a polygon is equal to as many times two right angles, as there are units in the number of sides diminished by two.



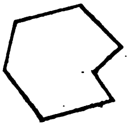
Let ABCDEFG be the proposed polygon. If from the vertex of any one angle A, diagonals AC, AD, AE, AF, be drawn to the vertices of all the opposite angles, it is plain that the polygon will be divided into five triangles, if it has seven sides; into six triangles, if it has eight;

and, in general, into as many triangles, all but two, as the polygon has sides; for those triangles may be considered as having the point A for a common vertex, and for bases, the several sides of the polygon, excepting the two which form the angle A. It is evident, also, that the sum of all the angles in those triangles does not differ from the sum of all the angles in the polygon: hence the latter sum is equal to as many times two right angles as there are triangles in the figure; in other words, as there are units in the number of sides diminished by two.

Cor. 1. The sum of the angles in a quadrilateral is equal to two right angles multiplied by 4—2, which amounts to four right angles: hence if all the angles of a quadrilateral are equal, each of them will be a right angle; a conclusion which sanctions our seventeenth Definition, where the four angles of a quadrilateral are asserted to be right, in the case of the rectangle and the square.

Cor. 2. The sum of the angles of a pentagon is equal to two right angles multiplied by 5—2, which amounts to six right angles: hence when a pentagon is equiangular, each angle is equal to the fifth part of six right angles, or to $\frac{6}{5}$ of one right angle.

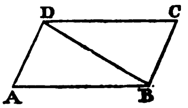
Cor. 3. The sum of the angles of a hexagon is equal to 2 × (6—2), or eight right angles: hence in the equiangular hexagon, each angle is the sixth part of eight right angles, or $\frac{4}{3}$ of one.



Scholium. When this proposition is applied to polygons which have *re-entrant* angles, each re-entrant angle must be regarded as greater than two right angles. But to avoid all ambiguity, we shall henceforth limit our reasoning to polygons with *salient* angles, which might otherwise be named *convex polygons.* Every convex polygon is such that a straight line, drawn at pleasure, cannot meet the contour of the polygon in more than two points.

PROPOSITION XXIX. THEOREM.

The opposite sides and angles of a parallelogram are equal.



Draw the diagonal BD. The triangles ADB, DBC have a common side BD; since AD, BC are parallel, they have also the angle ADB=DBC (Prop. 23.); and since AB, CD are parallel, the angle, ABD=BDC: hence they are equal (Prop. 7.); hence the side AB, opposite the angle ADB, is equal to the side DC, opposite the angle DBC; and in like manner, AD the third side is equal to BC: hence the opposite sides of a parallelogram are equal.

Again, since the triangles are equal, it follows that the angle A is equal to the angle C; and also that the angle ADC, composed of the two ADB, BDC, is equal to ABC, composed of the two DBC, ABD: hence the opposite angles of a parallelogram are equal.

Cor. Therefore two parallels AB, CD, included between two other parallels AD, BC, are equal.

PROPOSITION XXX. THEOREM.

If the opposite sides of a quadrilateral ABCD *are equal, namely,* AB=CD, *and* AD=BC, *the equal sides will be parallel, and the figure will be a parallelogram.*

For, having drawn the diagonal BD (see the preceding figure), the triangles ABD, BDC have all the sides of the one equal to the corresponding sides of the other; therefore they are equal; therefore the angle ADB, opposite the side AB, is equal to DBC, opposite CD; therefore (Prop. 23.) the side AD is parallel to BC. For a like reason, AB is parallel to CD: therefore the quadrilateral ABCD is a parallelogram.

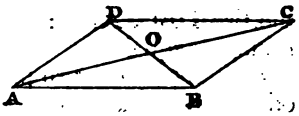
PROPOSITION XXXI. THEOREM.

If two opposite sides AB, CD *of a quadrilateral are equal and parallel, the remaining sides will also be equal and parallel, and the figure* ABCD *will be a parallelogram.*

Draw the diagonal BD (see the last figure). Since AB is parallel to CD, the alternate angles ABD, BDC are equal (Prop. 23.); moreover, the side BD is common, and the side AB=DC; hence the triangle ABD is equal (Prop. 6.) to DBC; hence the side AD is equal to BC, the angle ADB to DBC, and consequently AD is parallel to BC; hence the figure ABCD is a parallelogram.

PROPOSITION XXXII. THEOREM.

The two diagonals AC, DB *of a parallelogram divide each other into equal parts.*



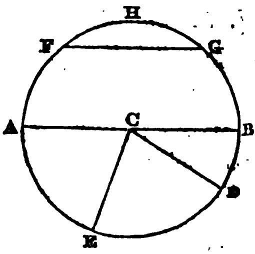
For, comparing the triangles ADO, COB, we find the side AD=CB, the angle ADO=CBO (Prop. 23.), and the angle DAO=OCB; hence (Prop. 7.) those triangles are equal; hence AO, the side opposite the angle ADO, is equal to OC opposite OBC; hence also DO is equal to OB.

Scholium. In the case of the rhombus, the sides AB, BC being equal, the triangles AOB, OBC have all the sides of the one equal to the corresponding sides of the other, and are therefore equal; whence it follows that the angles AOB, BOC are equal, and therefore, that the two diagonals of a rhombus cut each other at right angles.

BOOK II.

THE CIRCLE, AND THE MEASUREMENT OF ITS ANGLES.

Definitions.



I. The *circumference of a circle* is a curve line, all the points of which are equally distant from a point within, called the *centre.*

The *circle* is the space terminated by this curve line.

Note. In common language, the circle is sometimes confounded with its circumference: but the correct expression may always be easily recurred to, if we bear in mind that the circle is a surface which has length and breadth, while the circumference is but a line.

II. Every straight line, CA, CE, CD, drawn from the centre to the circumference, is called a *radius* or *semidiameter;* every line which, like AB, passes through the centre, and is terminated on both sides by the circumference, is called a *diameter.*

From the definition of a circle, it follows that all the radii are equal; that all the diameters are equal also, and each double of the radius.

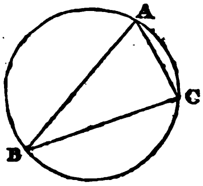
III. A portion of the circumference, such as FHG, is called an *arc*.

The *chord* or *subtense* of an arc is the straight line FG, which joins its two extremities.

IV. A *segment* is the surface, or portion of a circle, included between an arc and its chord.

Note. In all cases, the same chord FG belongs to two arcs, FHG, FEG, and consequently also to two segments: but the smaller one is always meant, unless the contrary is expressed.

V. A *sector* is the part of the circle included between an arc DE, and the two radii CD, CE, drawn to the extremities of the arc.

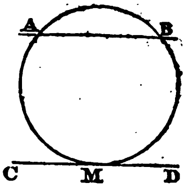


VI. A *line* is said to be *inscribed in a circle*, when its extremities are in the circumference, as AB.

An *inscribed angle* is one which, like BAC, has its vertex in the circumference, and is formed by two chords.

An *inscribed triangle* is one which, like BAC, has its three angular points in the circumference.

And, generally, an *inscribed figure* is one, of which all the angles have their vertices in the circumference. The circle is said to circumscribe such a figure.



VII. A *secant* is a line which meets the circumference in two points. AB is a secant.

VIII. A *tangent* is a line which has but one point in common with the circumference. CD is a tangent.

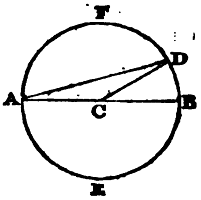
The point M is called the *point of contact*.

IX. In like manner, two circumferences *touch* each other when they have but one point in common.

X. A polygon is *circumscribed about a circle*, when all its sides are tangents to the circumference (see the diagram of Prop. 6. IV.): in the same case, the circle is said to be *inscribed* in the polygon.

PROPOSITION I. THEOREM.

Every diameter AB *divides the circle and its circumference into two equal parts.*



For, if the figure AEB be applied to AFB, their common base AB retaining its position, the curve line AEB must fall exactly on the curve line AFB, otherwise there would, in the one or the other, be points unequally distant from the centre, which is contrary to the definition of a circle.

PROPOSITION II. THEOREM.

Every chord is less than the diameter.

For, if the radii AC, CD (see the last figure) be drawn to the extremities of the chord AD, we shall have the straight line $AD \angle AC+CD$, or $AD \angle AB$.

Cor. Hence, the greatest line which can be inscribed in a circle is equal to its diameter.

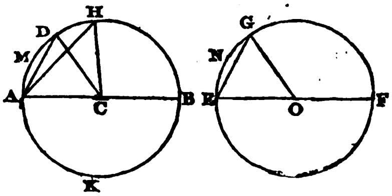
PROPOSITION III. THEOREM.

A straight line cannot meet the circumference of a circle in more than two points.

For, if it could meet it in three, those three points would be equally distant from the centre; and hence, there would be three equal straight lines drawn from the same point to the same straight line, which is impossible (Prop. 16. I.).

PROPOSITION IV. THEOREM.

In the same circle, or in equal circles, equal arcs are subtended by equal chords; and, conversely, equal chords subtend equal arcs.



If the radii AC, EO are equal, and the arcs AMD, ENG; then the chord AD will be equal to the chord EG.

For, since the diameters AB, EF are equal, the semicircle AMDB may be applied exactly to the semicircle ENGF, and the curve line AMDB will coincide entirely with the curve line ENGF. But the part AMD is equal to the part ENG *(Hyp.)*; hence the point D will fall on G; hence the chord AD is equal to the chord EG.

Conversely, supposing again the radii AC, EO to be equal, if the chord AD is equal to the chord EG, the arcs AMD, ENG will be equal.

For, if the radii CD, OG be drawn, the triangles ACD, EOG, having all their sides respectively equal, namely, AC=EO, CD=OG, and AD=EG, are themselves equal; and, consequently, the angle ACD is equal EOG. Now, placing the semicircle ADB on its equal EGF, since the angles ACD, EOG are equal, it is plain that the radius CD will fall on the radius OG, and the point D on the point G; therefore the arc AMD is equal to the arc ENG.

PROPOSITION V. THEOREM.

In the same circle, or in equal circles, a greater arc is subtended by a greater chord, and conversely,—the arcs being always supposed to be less than a semicircumference.

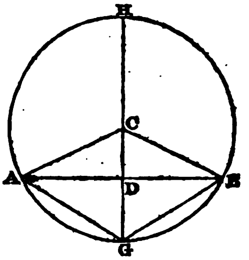
Let the arc AH be greater than AD (see the preceding figure); and draw the chords AD, AH, and the radii CD, CH. The two sides AC, CH of the triangle ACH are equal to the two AC, CD of the triangle ACD, and the angle ACH is greater than ACD; hence (Prop. 10. I.) the third side AH is greater than the third AD; hence the chord, which subtends the greater arc, is the greater.

Conversely, if the chord AH is greater than AD, it will follow, on comparing the same triangles, that the angle ACH is greater than ACD; and hence, that the arc AH is greater than AD.

Scholium. The arcs here treated of are each less than the semicircumference. If they were greater, the reverse property would have place; as the arcs increased, the chords would diminish, and conversely. Thus, the arc AKBD being greater than AKBH, the chord AD of the first is less than the chord AH of the second.

PROPOSITION VI. THEOREM.

The radius CG, *at right angles to a chord* AB, *divides it, and the subtended arc* AGB, *each into two equal parts.*



Draw the radii CA, CB. These radii considered with regard to the perpendicular CD, are two equal oblique lines; hence (Prop. 16. I.) they lie equally distant from that perpendicular: hence AD is equal to AD.

Again, since AD, DB are equal, CG is a perpendicular erected from the middle of AB; hence (Prop. 17. I.) every point of this perpendicular must be equally distant from its two extremities A and B. Now, G is one of those points; therefore AG, BG are equal. But, if the chord AG is equal to the chord GB, the arc AG will be equal to the arc GB; hence, the radius CG, at right angles to the chord AB, divides the arc subtended by that chord into two equal parts at the point G.

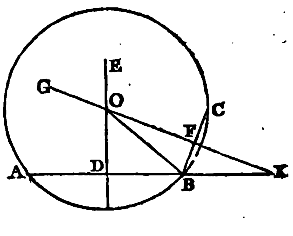
Scholium. The centre C, the middle point D of the chord AB, and the middle point G of the arc subtended by this chord, are three points situated in the same line perpendicular to the chord. But two points are sufficient to determine the position of a straight line; hence every straight line which passes through two of the points just mentioned, will necessarily pass through the third, and be perpendicular to the chord.

It follows, likewise, that *the perpendicular, raised from the middle of a chord, passes through the centre, and through the middle of the arc subtended by that chord.*

For this perpendicular is identical with the one let fall from the centre on the same chord, since both of them pass through the middle of the chord.

PROPOSITION VII. THEOREM.

Through three given points A, B, C, *not in the same straight line, one circumference may always be made to pass, and but one.*



Join AB, BC; and bisect those straight lines by the perpendiculars DE, FG: we assert first, that DE and FG, will meet in some point O.

For, they must necessarily cut each other, if they are not parallel. Now, suppose they were parallel, the line AB, which is perpendicular to DE, would also be perpendicular to FG (23. I.); and the angle K would be a right angle; but BK, the production of BD, is different from BF, because the three points A, B, C are not in the same straight line; hence there would be two perpendiculars, BF, BK, let fall from the same point on the same straight line, which (15. I.) is impossible; hence DE, FG will always meet in some point O.

And moreover, this point O, since it lies in the perpendicular DE, is equally distant from the two points, A and B (17. I.); the same point O, since it lies in the perpendicular FG, is also equally distant from the two points B and C: hence the three distances OA, OB, OC, are equal; hence the circumference described from the centre O, with the radius OB, will pass through the three given points A, B, C.

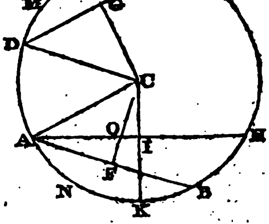
We have now shewn that one circumference can always be made to pass through three given points, not in the same straight line: we assert farther, that but one can be made to pass.

For, if there were a second circumference passing through the three given points A, B, C, its centre could not be out of the line DE (17. I.), for then it would be unequally distant from A and B; no more could it be out of the line FG, for a like reason; therefore, it would be in both the lines DE, FG. But two straight lines cannot cut each other in more than one point; hence there is but one circumference which can pass through three given points.

Cor. Two circumferences cannot meet in more than two points; for, if they have three common points, they must have the same centre, and form one and the same circumference.

PROPOSITION VIII. THEOREM.

Two equal chords are equally distant from the centre; and of two unequal chords, the less is farther from the centre.



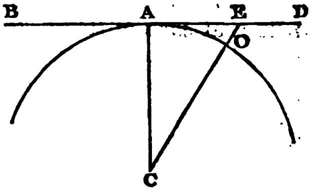
First. Suppose the chord AB=DE. Bisect those chords by the perpendiculars CF, CG, and draw the radii CA, CD.

In the right-angled triangles CAF, DCG, the hypotenuses CA, CD are equal; and the side AF, the half of AB, is equal to the side DG, the half of DE: hence the triangles are equal (18. I.), and CF is equal to CG; hence *(first)* the two equal chords AB, DE, are equally distant from the centre.

Secondly. Let the chord AH be greater than DE. The arc AKH (5. II.) will be greater than DME: cut off from the former, a part equal to the latter, ANB = DME; draw the chord AB, and let fall CF perpendicular to this chord, and CI perpendicular to AH. It is evident that CF is greater than CO, and CO than CI (16. I.); therefore, CF is still greater than CI. But CF is equal to CG, because the chords AB, DE, are equal; hence we have CG $>$ CI; hence of two unequal chords, the less is the farther from the centre.

PROPOSITION IX. THEOREM.

The straight BD, *perpendicular to the radius* CA, *and passing through its extremity, is a tangent to the circumference.*



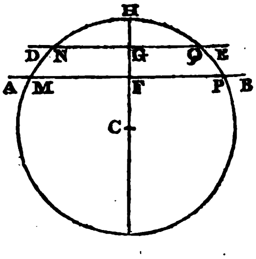
For (16. I.) every oblique line CE, is longer than the perpendicular CA; hence the point E is without the circle; hence BD has no point but A common to it with the circumference; hence BD (*Def.* 8. II.) is a tangent.

Scholium. From a given point A, no more than one tangent AD can be drawn to the circumference; for if another could be drawn, it would not be perpendicular to the radius CA; hence, in reference to this new tangent, the radius AC would be an oblique line, and the perpendicular let fall from the centre upon this tangent would be shorter than CA; hence this supposed tangent would enter the circle, and be a secant.

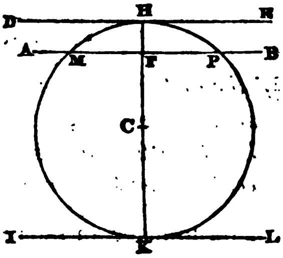
PROPOSITION X. THEOREM.

Two parallels AB, DE, *intercept equal arcs* MN, PQ, *on the circumference.*

There may be three cases.



First. If the two parallels are secants, draw the radius CH perpendicular to the chord MP. It will, at the same time, be perpendicular to NQ (23. I.); therefore, the point H (6. II.) will be at once the middle of the arc MHP, and of the arc NHQ; therefore, we shall have the arc MH=HP, and the arc NH=HQ; and therefore MH—NH=HP—HQ; in other words, MN=PQ.



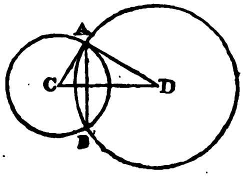
Second. When, of the two parallels AB, DE, one is a secant, the other a tangent, draw the radius CH to the point of contact H; it will be perpendicular to the tangent DE (9. II.), and also to its parallel MP. But, since CH is perpendicular to the chord MP, the point H must be the middle of the arc MHP; therefore the arcs MH, HP, included between the parallels AB, DE, are equal.

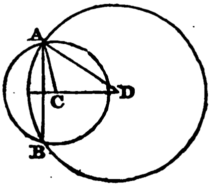
Third. If the two parallels DE, IL are tangents, the one at H, the other at K, draw the parallel secant AB; and, from what has just been shewn, we shall have MH=HP, MK=KP; and hence the whole arc HMK=HPK. It is farther evident that each of these arcs is a semicircumference.

PROPOSITION XI. THEOREM.

If two circles cut each other in two points, the line which passes through their centres, will be perpendicular to the chord which joins the points of intersection, and will divide it into two equal parts.

For the line AB, which joins the points of intersection, is a chord common to the two circles. And if a perpendicular be erected from the middle of this chord, it will pass (6. II.) through each of the two centres C and D. But no more than one straight line can be drawn through two points; hence the straight line, which passes through the centres, will bisect the chord at right angles.





PROPOSITION XII. THEOREM.

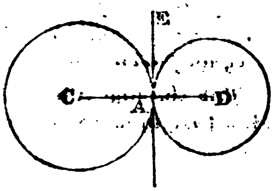
If the distance between the centres of two circles is less than the sum of the radii, the greater radius being at the same time less than the sum of the smaller and the distance between the centres, the two circles will cut each other.

For to make an intersection possible, the triangle CAD (see the preceding figure) must be possible. Hence, not only must we have $CD \angle AC + AD$, but also the greater radius $AD \angle AC + CD$. And, whenever the triangle CAD can be constructed, it is plain that the circles described from the centres C and D, will cut each other in A and B.

PROPOSITION XIII. THEOREM.

If the distance CD, *between the centres of two circles is equal to the sum of their radii* CA, AD, *those two circles will touch each other externally.*

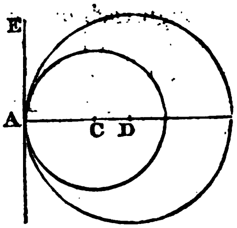
They will evidently have the point A common, and they will have no other; because, if they had two points common, the distance between their centres must be less than the sum of their radii.



PROPOSITION XIV. THEOREM.

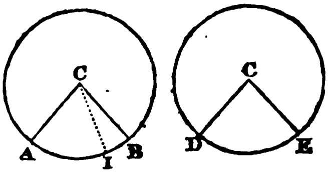
If the distance CD, *between the centres of two circles, is equal to the difference of their radii* CA, AD, *those two circles will touch each other externally.*

It is evident, as before, that they will have the point A common: they can have no other; because, if they had, the greater radius AD (12. II.) must be less than the sum of the radius AC and the distance between the centres, which it is not.



PROPOSITION XV. THEOREM.

In the same circle, or in equal circles, the equal angles ACB, DCE, *having their vertices in the centre, intercept equal arcs* AB, DE, *on the circumference: and conversely, if the arcs* AB, DE *are equal, the angles* ACB, DCE *will also be equal.*

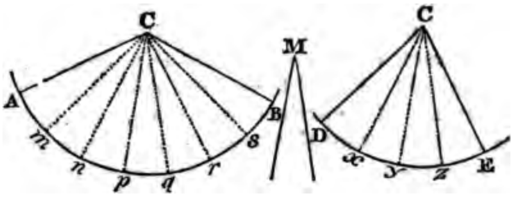


First. Since the angles ACB, DCE are equal, they may be placed upon each other; and since their sides are equal, the point A will evidently fall on D, and the point B on E. But, in that case, the arc AB must also fall on the arc DE; for if they did not exactly coincide, there would, in the one or the other, be points unequally distant from the centre; which is impossible: hence the arc AB is equal to DE.

Secondly. If we suppose AB=DE, the angle ACB will be equal to DCE. For if those angles are not equal, let ACB be the greater, and let ACI be taken equal to DCE. From what has just been shewn, we shall have AI=DE: but, by hypothesis, AB is equal to DE; hence AI must be equal to AB, the part to the whole, which is absurd: hence the angle ACB is equal to DCE.

PROPOSITION XVI. THEOREM.

In the same circle, or in equal circles, if two angles at the centre ACB, DCE *are to each other in the proportion of two whole numbers, the intercepted arcs* AB, DE *will be to each other in the proportion of the same numbers, and we shall have the angle* ACB : *angle* DCE : : *arc* AB : *arc* DE.

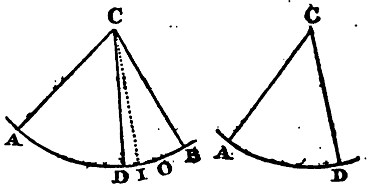


Suppose, for example, that the angles ACB, DCE are to each other as 7 is to 4; or, which is the same thing, suppose that the angle M, which may serve as a common measure, is contained 7 times in the angle ACB and 4 times in DCE. The seven partial angles ACm, mCn, nCp, &c., into which ACB is divided, being each equal to any of the four partial angles into which DCE is divided; each of the partial arcs Am, mn, np, &c., will (15. II.) be equal to each of the partial arcs Ax, xy, &c. Therefore the whole arc AB will be to the whole arc DE as 7 is to 4. But the same reasoning would evidently apply, if in place of 7 and 4 any numbers whatever were employed; hence, if the ratio of the angles ACB, DCE can be expressed in whole numbers, the arcs AB, DE will be to each other as the angles ACB, DCE.

Scholium. Conversely, if the arcs AB, DE are to each other as two whole numbers, the angles, ACB, DCE will be to each other as the same whole numbers, and we shall have $ACB : DCE :: AB : DE$. For the partial arcs, Am, mn, &c. and Dx, xy, &c. being equal, the partial angles ACm, mCn, &c. and DCx, xCy, &c. will also be equal.

PROPOSITION XVII. THEOREM.

Whatever be the ratio of the two angles ACB, ACD, *those two angles will always be to each other as the arcs* AB, AD *intercepted between their sides, and described from their vertices as centres, with equal radii.*



Let the less angle be placed on the greater. If the proposition is not true, the angle ACB will be to the angle ACD as the arc AB is to an arc greater or less than AD. Suppose this arc to be greater, and let it be represented by AO; we shall thus have the angle ACB : angle ACD :: arc AB : arc AO. Next conceive the arc AB to be divided into equal parts, each of which is less than DO; there will be at least one point of division between D and O; let I be that point; and join CI. The arcs AB, AI will be to each other as two whole numbers, and by the preceding theorem, we shall have the angle ACB : angle ACI :: arc AB : arc AI. Comparing these two proportions with each other, and observing that the antecedents are the same, we infer that the consequents are proportional, and thus we find the angle ACD : angle ACI :: arc AO : arc AI. But the arc AO is greater than the arc AI; hence, if this proportion is true, the angle ACD must be greater than the angle ACI: on the contrary, however, it is less; hence the angle ACB cannot be to the angle ACD as the arc AB is to an arc greater than AD.

By a process of reasoning entirely similar, it may be shewn that the fourth term of the proportion cannot be less than AD; hence it is AD itself; hence we have

Angle ACB : angle ACD : : arc AB : arc AD.

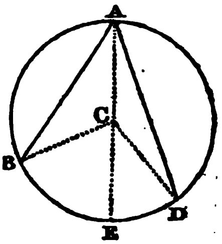
Cor. Since the angle at the centre of a circle, and the arc intercepted by its sides, have such a connexion, that if the one be augmented or diminished in any ratio, the other will be augmented or diminished in the same ratio, we are authorised to establish the one of those magnitudes as the measure of the other; and we shall henceforth assume the arc AB as the measure of the angle ACB. It is only required that, in the comparison of angles with each other, the arcs which serve to measure them, be described with equal radii, as all the foregoing propositions imply.

Scholium 1. It appears most natural to measure a quantity by a quantity of the same species; and upon this principle it would be convenient to refer all angles to the right angle; which, being made the unit of measure, an acute angle would be expressed by some number between 0 and 1; an obtuse angle by some number between 1 and 2. This mode of expressing angles would not, however, be the most convenient in practice; it has been found more simple to measure them by arcs of a circle, on account of the facility with which arcs can be made equal to given arcs, and for various other reasons. At all events, if the measurement of angles by arcs of a circle is in any degree indirect, it is still equally easy to obtain the direct and absolute measure by this method; since, on comparing the arc which serves as a measure to any angle, with the fourth part of the circumference, we find the ratio of the given angle to a right angle, which is the absolute measure.

Scholium 2. All that has been demonstrated in the last three propositions, concerning the comparison of angles with arcs, holds true equally, if applied to the comparison of sectors with arcs; for sectors are not only equal when their angles are so, but in all respects proportional to their angles; hence *two sectors* ACB, ACD, *taken in the same circle, or in equal circles, are to each other as the arcs* AB, AD, *the bases of those sectors.* It is hence evident that the arcs of the circle, which serve as a measure of the different angles, may also serve as a measure of the different sectors, in the same circle, or equal circles.

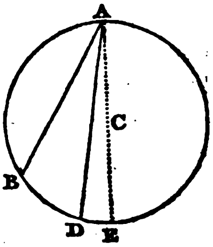
PROPOSITION XVIII. THEOREM.

The inscribed angle BAD *is measured by half the arc* BD, *included between its sides.*



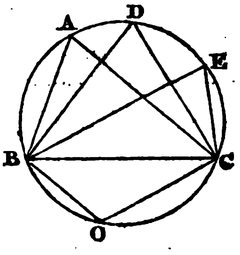
Let us first suppose that the centre of the circle lies within the angle BAD. Draw the diameter AE, and the radii CB, CD.

The angle BCE, being exterior to the triangle ABC, is equal to the sum of the two interior angles CAB, ABC (23.1): but the triangle BAC being isosceles, the angle CAB is equal to ABC; hence the angle BCE is double of BAC. Since BCE lies at the centre, it is measured by the arc BE; hence BAC will be measured by the half of BE. For a like reason, the angle CAD will be measured by the half of ED; hence BAC+CAD or BAD will be measured by the half of BE + ED or of BD.

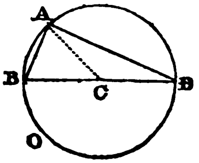


Suppose, in the second place, that C the centre lies without the angle BAD. Then, drawing the diameter AE, the angle BAE will be measured by the half of BE; the angle DAE by the half of DE: hence their difference BAD will be measured by the half of BE *minus* the half of ED, or by the half of BD.

Hence every inscribed angle is measured by the half of the arc included between its sides.



Cor. 1. All the angles BAC, BDC, inscribed in the same segment are equal; because they are all measured by the half of the same arc BOC.



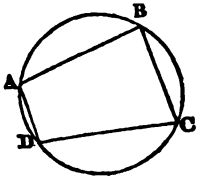
Cor. 2. Every angle BAD, inscribed in a semicircle, is a right angle; because it is measured by half the semicircumference BOD, that is, by the fourth part of the whole circumference.

To demonstrate the same property another way, draw the radius AC: the triangle BAC is isosceles, hence the angle BAC=ABC; the triangle CAD is also isosceles, hence the angle CAD=ADC; hence BAC+CAD, or BAD=ABD+ADB. But if the two angles B and D of the triangle ABD are together equal to the third BAD, all the three angles will be together equal to twice BAD; we already know that they are equal to two right angles; therefore BAD is equal to one.

Cor. 3. Every angle BAC (see the diagram of Cor. 1.) inscribed in a segment greater than a semicircle, is an acute angle; for it is measured by the half of the arc BOC, less than a semicircumference.

And every angle BOC, inscribed in a segment less than a semicircle, is an obtuse angle; for it is measured by the half of the arc BAC, greater than a semicircumference.

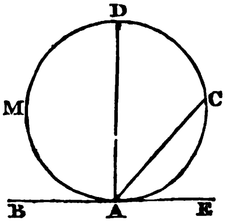
Cor. 4. The opposite angles A and C, of an inscribed quadrilateral ABCD, are together equal to two right angles: for the angles BAD is measured by half the arc BCD, the angle BCD is measured by half the arc BAD; hence the two angles BAD, BCD, taken together, are measured by the half of the circumference; hence their sum is equal to two right angles.



PROPOSITION XIX. THEOREM.

The angle BAC, *formed by a tangent and a chord, is measured by the half of the arc* AMDC, *included between its sides.*

From A, the point of contact, draw the diameter AD. The angle BAD is right (9. II.) and is measured by half the circumference AMD; the angle DAC is measured by the half of DC: hence BAD+DAC, or BAC is measured by the half of AMD plus the half of DC, or by half the whole arc AMDC.

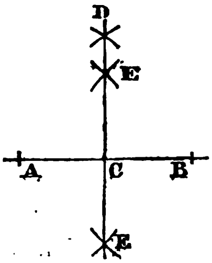


It might be shewn, in the same manner, that the angle CAE is measured by half the arc AC included between its sides.

PROBLEMS RELATING TO THE FIRST TWO BOOKS.

PROBLEM I.

To divide the given straight line AB *into two equal parts.*

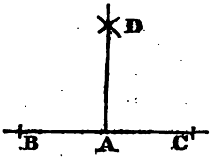


From the points A and B as centres, with a radius greater than the half of AB, describe two arcs cutting each other in B; the point B will be equally distant from A and B. Find, in like manner, above or beneath the line AB, a second point E, equally distant from the points A and B; through the two points D and E, draw the line DE: it will bisect the line AB in C.

For the two points D and E, being each equally distant from the extremities A and B, must both lie in the perpendicular raised from the middle of AB. But only one straight line can pass through two given points; hence the line DE must itself be that perpendicular, which cuts AB into two equal parts at the point C.

PROBLEM II.

At a given point A, *in a given straight line* BC, *to erect a perpendicular to that line.*



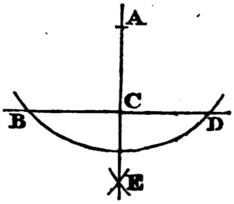
Take the points B and C at equal distances from A; then from the points B and C as centres, with a radius greater than BA, describe two arcs intersecting each other in D; draw AD: it will be the perpendicular required.

For the point D, being equally distant from B and C, belongs to the perpendicular raised from the middle of BC; therefore AD is that perpendicular.

Scholium. The same construction serves for making a right angle BAD, at a given point, A, on a given straight line BC.

PROBLEM III.

From a given point A, *without the straight line* BD, *to let fall a perpendicular on that line.*

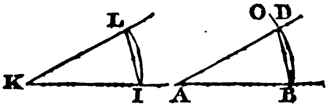


From the point A as a centre, and with a radius sufficiently great, describe an arc cutting the line BD in the two points B and D; then mark a point E, equally distant from the points B and D, and draw AE: it will be the perpendicular required.

For, the two points A and E are each equally distant from the points B and D; hence the line AE is a perpendicular passing through the middle of BD.

PROBLEM IV.

At the point A *in the line* AB, *to make an angle equal to the given angle* K.

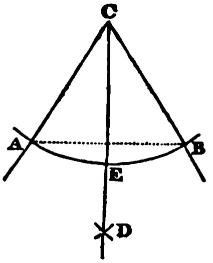


From the vertex K as a centre, with any radius, describe the arc IL, terminating in the two sides of the angle; from the point A as a centre, with a distance AB equal to KI, describe the indefinite arc BO; then take a radius equal to the chord LI, with which, from the point B as a centre, describe an arc cutting the indefinite one BO, in D; draw AD; and the angle DAB will be equal to the given angle K.

For the two arcs BD, LI have equal radii, and equal chords; hence they are equal (4. II.); hence the angles BAD, IKL measured by them are equal.

PROBLEM V.

To divide a given angle, or a given arc, into two equal parts.



First. Let it be required to divide the arc AB into two equal parts. From the points A and B as centres, with the same radius, describe two arcs cutting each other in D; through the point D and the centre C, draw CD: it will bisect the arc AB in the point E.

For the two points C and D are each equally distant from the extremities A and B of the chord AB; hence the line CD bisects the chord at right angles; hence (6. II.) it bisects the arc AB in the point E.

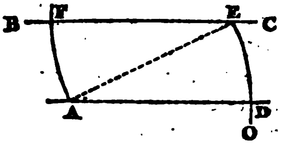
Secondly. Let it be required to divide the angle ACB into two equal parts. We begin by describing, from the vertex C as a centre, the arc AB; which is then bisected as above. It is plain that the line CD will divide the angle ACB into two equal parts.

Scholium. By the same construction, each of the halves AE, EB may be divided into two equal parts; and thus by successive subdivisions, a given angle or a given arc may be divided into four equal parts, into eight, into sixteen, and so on.

PROBLEM VI.

Through a given point A, *to draw a parallel to a given straight line* BC.

From the point A as a centre, with a radius sufficiently great, describe the indefinite arc EO; from the point E as a centre, with the same radius, describe the arc AF; take EO=AF, and draw AD: this will be the parallel required.

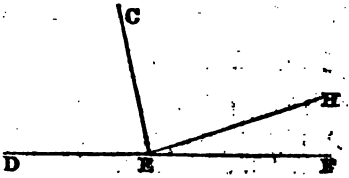


For, joining AE, the alternate angles AEF, EAD are evidently equal; therefore (23. I.) the lines AD, EF are parallel.

PROBLEM VII.

Two angles A *and* B *of a triangle being given, to find the third.*

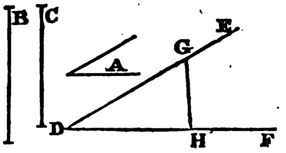
Draw the indefinite line DEF; at the point E, make the angle DEC = A, and the angle CEH=CB: the remaining angle HEF will be the third angle required; because those three angles are together equal to two right angles.



PROBLEM VIII.

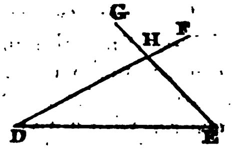
Two sides B *and* C *of a triangle, and the angle* A *which they contain, being given, to describe the triangle.*

Having drawn the indefinite line DE; at the point D, make the angle EDF equal to the given angle A; then take DG=B, DH=C, and draw GH; DGH will be the triangle required.



PROBLEM IX.

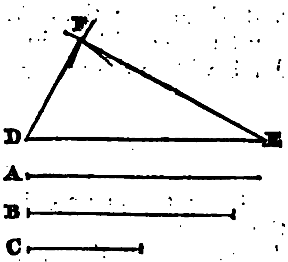
A side and two angles of a triangle being given, to describe the triangle.



The two angles will either be both adjacent to the given side, or the one adjacent and the other opposite: in the latter case, find the third angle (Prob. 7.); and the two adjacent angles will thus be known: in the former case, draw the straight line DE equal to the given side; at the point D, make an angle EDF equal to one of the adjacent angles, and at E, an angle DEG equal to the other; the two lines DF, EG, will cut each other in H; and DEH will be the triangle required.

PROBLEM X.

The three sides A, B, C *of a triangle being given, to describe the triangle.*

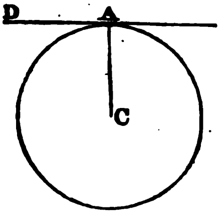


Draw DE equal to the side A; from the point E as a centre with a radius equal to the second side B, describe an arc; from D as a centre with a radius equal to the third side C, describe another arc intersecting the former in F; draw DF, EF; and DEF will be the triangle required.

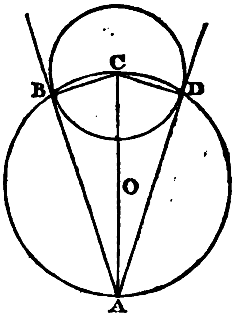
Scholium. If one of the sides were greater than the sum of the other two, the arcs would not intersect each other: but the solution will always be possible, when the sum of two sides, anyhow taken, is greater than the third.

PROBLEM XIV.

Through a given point to draw a tangent to a given circle.



If the given point A lies in the circumference, draw the radius CA, and erect AD perpendicular to it: AD (9. II.) will be the tangent required.



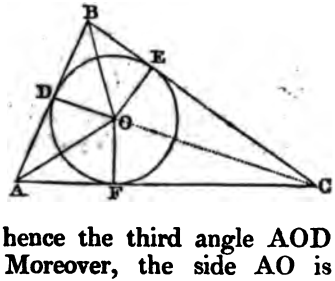
If the point A lies without the circle, join A and the centre by the straight line CA; bisect CA in O; from O as a centre, with the radius OC, describe a circle intersecting the given circumference in B; join AB: this will be the tangent required.

For, drawing CB, the angle CBA being inscribed in a semicircle is a right angle (18. II.); therefore AB is a perpendicular at the extremity of the radius CB; therefore it is a tangent.

Scholium. When the point A lies without the circle, there will evidently be always two equal tangents AB, AD, passing through the point A: they are equal, because the right-angled triangles CBA, CDA have the hypotenuse CA common, and the side CB=CD; hence they are equal; hence AD is equal to AB, and also the angle CAD to CAB.

PROBLEM XV.

To inscribe a circle in a given triangle ABC.



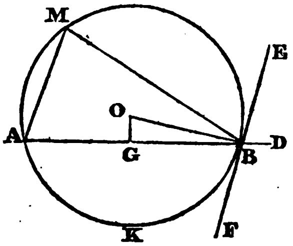
Bisect the angles A and B, by the lines AO and BO, meeting in the point O; from the point O, let fall the perpendiculars OD, OE, OF, on the three sides of the triangle: these perpendiculars will all be equal. For, by construction, we have the angle DAO=OAF, the right angle ADO=AFO; hence the third angle AOD is equal to the third AOF. Moreover, the side AO is common to the two triangles AOD, AOF; and the angles adjacent to the equal side are equal: hence the triangles themselves are equal; and DO is equal to OF. In the same manner it may be shewn that the two triangles BOD, BOE are equal; therefore OD is equal to OC; therefore the three perpendiculars OD, OE, OF are all equal.

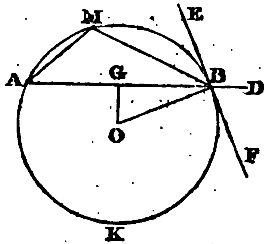
Now, if from the point O as a centre, with the radius OD, a circle be described, this circle will evidently be inscribed in the triangle ABC; for the side AB, being perpendicular to the radius at its extremity, is a tangent; and the same thing is true of the sides BC, AC.

Scholium. The three lines which bisect the angles of a triangle meet in the same point.

PROBLEM XVI.

On a given straight line AB, *to describe a segment containing a given angle* C, *that is to say, a segment such that all the angles inscribed in it shall be equal to the given angle* C.





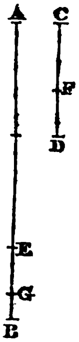
Produce AB towards D; at the point B, make the angle $DBE=C$; draw BO perpendicular to BE, and GO perpendicular to AB, and bisecting it; and from the point O, where those perpendiculars meet as a centre, with a distance OB describe a circle: the required segment will be AMB.

For since BF is a perpendicular at the extremity of the radius OB, it is a tangent, and the angle ABF (19. II.) is measured by half the arc AKB. Also the angle AMB, being an inscribed angle, is measured by half the arc AKB: hence we have $AMB=ABF=EBD=C$: hence all the angles inscribed in the segment AMB are equal to the given angle C.

Scholium. If the given angle were right, the required segment would be the semicircle described on the diameter AB.

PROBLEM XVII.

To find the numerical ratio of two given straight lines AB, CD, *those lines being supposed to have a common measure.*



From the greater AB cut off a part equal to the less CD, as many times as possible; for example, twice, with the remainder BE.

From the line CD, cut off a part equal to the remainder BE, as many times as possible; once, for example, with the remainder DF.

From the first remainder BE, cut off a part equal to the second DF, as many times as possible; once, for example, with the remainder BG.

From the second remainder DF, cut off a part equal to BG the third, as many times as possible.

Continue this process, till a remainder occur, which is contained exactly a certain number of times in the preceding one.

Then this last remainder will be the common measure of the proposed lines; and regarding it as unity, we shall easily find the values of the preceding remainders; and, at last, those of the two proposed lines, and hence their ratio in numbers.

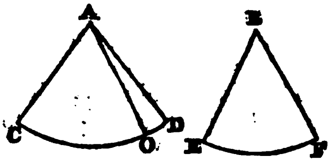
Suppose, for instance, we find GB to be contained exactly twice in FD; BG will be the common measure of the two proposed lines. Put $BG=1$; we shall have $FD=2$: but EB contains FD once, *plus* GB; therefore we have $EB=3$: CD contains EB once, *plus* FD; therefore we have $CD=5$: and, lastly, AB contains CD twice, *plus* EB; therefore we have $AB=13$; hence the ratio of the lines is that of 13 to 5. If the line CD were taken for unity, the line AB would be $\frac{13}{5}$; if AB were taken for unity, CD would be $\frac{5}{13}$.

Scholium. The method just explained is the same as that employed in arithmetic to find the common divisor of two numbers: it has no need, therefore, of any other demonstration.

How far soever the operation be continued, it is possible that no remainder may ever be found, which shall be contained an exact number of times in the preceding one. When this happens, the two lines have no common measure, and are said to be *incommensurable.* An instance of this will be seen afterwards, in the ratio of the diagonal to the side of the square. In those cases, therefore, the exact ratio in numbers cannot be found: but, by neglecting the last remainder, an approximate ratio will be obtained, more or less correct, according as the operation has been continued a greater or less number of times.

PROBLEM XVIII.

Two angles A and B being given, to find their common measure, if they have one, and by means of it their ratio in numbers.



With equal radii describe the arcs CD, EF, to serve as measures for the angles: proceed afterwards in the comparison of the arcs CD, EF, as in the last problem, since an arc may be cut off from an arc of the same radius, as a straight line from a straight line. We shall thus arrive at the common measure of the arcs CD, EF, if they have one, and thereby at their ratio in numbers. This ratio (17. II.) will be the same as that of the given angles; and if DO is the common measure of the arcs, DAO will be that of the angles.

Scholium. According to this method, the absolute value of an angle may be found by comparing the arc which measures it to the whole circumference. If the arc CD, for example, is to the circumference as 3 is to 25, the angle A will be $\frac{3}{25}$ of four right angles, or $\frac{12}{25}$ of one right angle.

It may also happen that the arcs compared have no common measure; in which case, the numerical ratios of the angles will only be found approximately with more or less correctness, according as the operation has been continued a greater or less number of times.

BOOK III.

THE PROPORTIONS OF FIGURES.

Definitions.

I. We shall give the name *equivalent figures* to such as have equal surfaces.

Two figures may be equivalent though very dissimilar: a circle, for instance, may be equivalent to a square, a triangle to a rectangle.

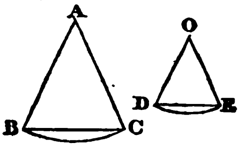
The denomination, *equal figures*, we shall reserve for such as, when applied to each other, coincide in all their points: of this kind are two circles, which have equal radii; two triangles, which have all their sides equal respectively, &c.

II. Two figures are *similar*, when they have their angles equal each to each, and their *homologous sides* proportional. By homologous sides, are understood those which have a corresponding position in the two figures, or which lie adjacent to equal angles. Those angles themselves are called *homologous angles*.

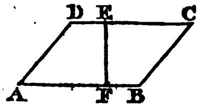
Two equal figures are always similar; but two similar figures may be very unequal.

III. In two different circles, *similar arcs*, *sectors*, *segments*, are those which correspond to equal angles at the centre.

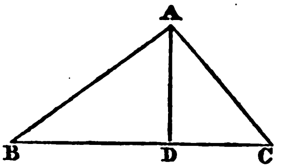
Thus, if the angles A and O be equal, the arc BC will be similar to the arc DE, the sector ABC to the sector ODE, &c.



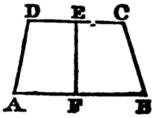
IV. The *altitude* of a parallelogram is the perpendicular EF which measures the distance of two opposite sides AB, CD, taken as bases.



V. The *altitude* of a triangle is the perpendicular AD let fall from the vertex of an angle A, on the opposite side BC taken as a base.



VI. The *altitude* of a trapezium is the perpendicular EF drawn between its two parallel sides AB, CD.



VII. The *area* and the surface of a figure are terms very nearly synonimous. The area designates more particularly the superficial extent of the figure, in so far as it is measured, or compared to other surfaces.

Note. For understanding this Book, and those that follow, the reader will require to be master of the theory of proportions, for which we refer him to the common treatises on arithmetic and algebra.* We shall only make one remark, which is of great importance for determining the true meaning of propositions, and dissipating any obscurity that may occur either in the enunciations or the proofs.

The proportion A : B : : C : D being given, it is well known that the product of the extremes A×D is equal to the product of the means B×C.

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* A short sketch of the subject has been prefixed to the present translation.——Ed.

This truth is indisputable, so far as concerns numbers: it is equally so in regard to magnitudes of any kind, provided they are expressed, or imagined to be expressed, in numbers; and this we are at all times entitled to imagine. If A, B, C, D, for example, are lines, we may conceive that one of those four lines, or a fifth, if requisite, serves as a common measure to them all, and is taken for unity: then A, B, C, D, represent each a certain number of units, integer or fractional, commensurable or incommensurable; and the proportion between the lines A, B, C, D, becomes a proportion of numbers.

The product of the lines A and D, which is also named their *rectangle*, is therefore nothing else than the number of linear units contained in A, multiplied by the number of linear units contained in D; and it is easy to see that this product may, indeed must, be equal to that which results in a similar manner from the lines B and C.

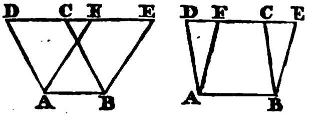
The magnitudes A and B may be of one species, for example, lines; and C and D of another species, for example, surfaces: in such cases, those magnitudes must always be regarded as numbers; A and B will be expressed in linear units, C and D in superficial units; and the product A×D will be a number like the product B×C.

Generally, in every operation connected with proportions, if the terms of those proportions be always looked upon as so many numbers, each of the kind proper to it, there will be no difficulty in conceiving the operations, and the consequences which result from them.

We must also premise, that several of our demonstrations are grounded on some of the simpler operations of algebra, which are themselves dependent on admitted axioms. Thus, if we have $A=B+C$, and if each member is multiplied by the same quantity M, we may infer that $A\times M=B\times M+C\times M$; in like manner, if we have $A=B+C$, and $D=E-C$, and if the equal quantities are added together, then expunging the $+C$ and $-C$, which destroy each other, we infer that $A+D=B+E$, and so of others All this is evident enough of itself; but in case of difficulty, it will be useful to consult some algebraical treatise, and thus to combine the study of the two sciences.

PROPOSITION I. THEOREM.

Parallelograms which have equal bases and equal altitudes are equivalent.



Let AB be the common base of the two parallelograms ABCD, ABEF: and since they are supposed to have the same altitude, their upper bases DC, FE will be situated both in one straight line parallel to AB. Now, from the nature of parallelograms, we have AD=BC, and AF=BE; for the same reason, we have DC=AB, and FE=AB; hence DC=FE: hence, if DC and FE be taken away from the same line DE, the remainders CE and DF will be equal.

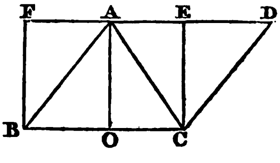
It follows (11. I.) that the triangles DAF, CBD, are mutually equilateral, and consequently equal.

But if from the quadrilateral ABED, we take away the triangle ADF, there will remain the parallelogram ABEF: and if from the same quadrilateral ABED, we take away the triangle CBE, there will remain the parallelogram ABCD. Hence these two parallelograms ABCD, ABEF, which have the same base and altitude, are equivalent.

Cor. Every parallelogram is equivalent to the rectangle which has the same base and the same altitude.

PROPOSITION II. THEOREM.

Every triangle ABC *is half of the parallelogram* ABCD, *which has the same base and the same altitude.*



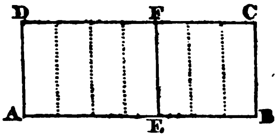
For (31. I.), the triangles ABC, ACD are equal.

Cor. 1. Hence a triangle ABC is half of the rectangle BCEF, which has the same base CB, and the same altitude AO: for the rectangle BCEF is equivalent to the parallelogram ABCD.

Cor. 2. All triangles, which have equal bases and altitudes, are equivalent.

PROPOSITION III. THEOREM.

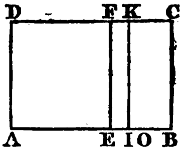
Two rectangles having the same altitude are to each other as their bases.



Let ABCD, AEFD be two rectangles having the common altitude AD: they are to each other as their bases AB, AE.

Suppose, first, that the bases are commensurable, and are to each other, for example, as the numbers 7 and 4. If AB is divided into 7 equal parts, AE will contain 4 of those parts: at each point of division erect a perpendicular to the base; seven partial rectangles will thus be formed, all equal to each other, because all have the same base and altitude. The rectangle ABCD will contain seven partial rectangles, while AEFD will contain four: hence the rectangle ABCD is to AEFD as 7 is to 4, or as AB is to AE. The same reasoning may be applied to any other ratio equally with that of 7 to 4: hence, whatever be that ratio, if its terms be commensurable, we shall have

ABCD : AEFD : : AB : AE.



Suppose, in the second place, that the bases AB, AE are incommensurable: it is to be shewn that still we shall have

ABCD : AEFD : : AB : AE.

For if not, the first three terms continuing the same, the fourth must be greater or less than AE. Suppose it to be greater, and that we have

ABCD : AEFD : : AB : AO.

Divide the line AB into equal parts each less than EO. There will be at least one point I of division between E and O: from this point draw IK perpendicular to AI: the bases AB, AI will be commensurable, and thus, from what is proved above, we shall have

ABCD : AIKD : : AB : AI.

But by hypothesis, we have

ABCD : AEFD : : AB : AO.

In these two proportions the antecedents are equal; hence the consequents are proportional, and we find

AIKD : AEFD : : AI : AO.

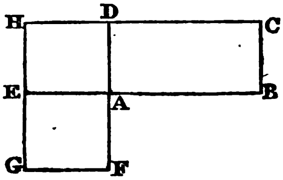
But AO is greater than AI; hence, if this proportion is correct, the rectangle AEFD must be greater than AIKD: on the contrary, however, it is less; hence the proportion is impossible; hence ABCD cannot be to AEFD as AB is to a line greater than AE.

Exactly in the same manner, it may be shewn that the fourth term of the proportion cannot be less than AE; therefore it is equal to AE.

Hence, whatever be the ratio of the bases, two rectangles ABCD, AEFD, of the same altitude, are to each other as their bases AB, AE.

PROPOSITION IV. THEOREM.

Any two rectangles ABCD, AEGF, *are to each other as the products of the bases multiplied by the altitudes, so that we have* ABCD : AEGF : : AB × AD : AE × AF.



Having placed the two rectangles, so that the angles at A are vertical, produce the sides GE, CD till they meet in H. The two rectangles ABCD, AEHD, having the same altitude, are to each other as their bases AB, AE: in like manner the two rectangles AEHD, AEGF, having the same altitude AE, are to each other as their bases AD, AF: thus we have the two proportions,

ABCD : AEHD : : AB : AE,

AEHD : AEGF : : AD : AF.

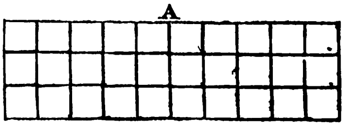
Multiplying the corresponding terms of those proportions together, and observing that the mean term AEHD may be omitted, since it is a multiplier common to both the antecedent and the consequent, we shall have

ABCD : AEGF : : AB × AD : AE × AF.

Scholium. Hence the product of the base by the altitude may be assumed as the measure of a rectangle, provided we understand by this product, the product of two numbers, one of which is the number of linear units contained in the base, the other the number of linear units contained in the altitude.

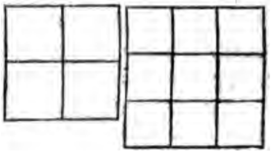
Still this measure is not absolute but relative: it supposes that the area of any other rectangle is computed in a similar manner, by measuring its sides with the same linear unit; a second product is thus obtained, and the ratio of the two products is the same as that of the rectangles, agreeably to the proposition just demonstrated.

For example, if the base of the rectangle A contains three units, and its altitude ten, that rectangle will be represented by the number 3×10, or 30, a number which signifies nothing while thus isolated; but if there is a second rectangle B, the base of which contains twelve units, and the altitude seven, this second rectangle will be represented by the number $12 \times 7 = 84$; and we shall hence be entitled to conclude that the two rectangles are to each other as 30 is to 84; and therefore, if the rectangle A were to be assumed as the unit of measurement in surfaces, the rectangle B would then have $\frac{84}{30}$ for its absolute measure, in other words, it would be equal to $\frac{84}{30}$ of a superficial unit.



It is more common and more simple, to assume the square as the unit of surface; and to select that square, whose side is the unit of length. In this case, the measurement which we have regarded merely as relative, becomes absolute: the number 30, for instance, by which the rectangle A was measured, now represents 30 superficial units, or 30 of those squares, which have each of their sides equal to unity, as the diagram exhibits.

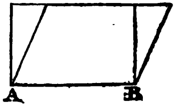
In geometry the product of two lines is frequently confounded with their *rectangle*, and this expression has passed into arithmetic, where it serves to designate the product of two unequal numbers, the expression *square* being employed to designate the product of a number multiplied by itself.



The arithmetical squares of 1, 2, 3, &c. are 1, 4, 9, &c. So likewise the geometrical square constructed on a double line is evidently four times as great as on a single one; on a triple line, is nine times as great, &c.

PROPOSITION V. THEOREM.

The area of any parallelogram is equal to the product of its base by its altitude.

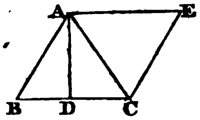


For the parallelogram ABCD is equivalent (1. III. Cor.) to the rectangle ABEF, which has the same base AB, and the same altitude BE: but this rectangle (4. III. Schol.) is measured by $AB \times BE$; therefore $AB \times BE$ is equal to the area of the parallelogram ABCD.

Cor. Parallelograms of the same base are to each other as their altitudes; and parallelograms of the same altitude are to each other as their bases: for if A, B, C are any three magnitudes, we have always $A\times C : B\times C :: A : B$.

PROPOSITION VI. THEOREM.

The area of a triangle is equal to the product of its base by half its altitude.

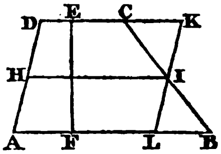


For, the triangle ABC (2. III.) is half of the parallelogram ABCE, which has the same base BC, and the same altitude AD: but the surface of the parallelogram (5. III.) is equal to $BC\times AD$; hence that of the triangle must be $\frac{1}{2}BC\times AD$, or $BC\times{}^{1}AD$.

Cor. Two triangles of the same altitude are to each other as their bases, and two triangles of the same base are to each other as their altitudes.

PROPOSITION VII. THEOREM.

The area of the trapezium ABCD *is equal to its altitude* EF, *multiplied by the half sum of its parallel bases,* AB, CD.



Through I, the middle point of the side BC, draw KL parallel to the opposite side AD; and produce DC till it meet KL.

In the triangles IBL, ICK, we have the side IB=IC, by construction; the angle LIB=CIK; and (23. I.) since CK and BL are parallel, the angle IBL=ICK; hence the triangles are equal; hence the trapezium ABCD is equivalent to the parallelogram ADKL, and is measured by $EF\times AL$.

But we have AL=DK; and since the triangles IBL and KCI are equal, the side BL=CK: hence $AB+CD=AL+DK=2AL$; hence AL is the half sum of the bases AB, CD; hence the area of the trapezium ABCD is equal to the altitude EF multiplied by the half sum of the bases AB, CD, a result which is expressed thus: $ABCD=EF\times\frac{AB+CD}{2}$.

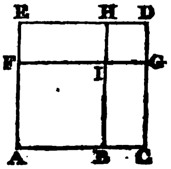
Scholium. If through I, the middle point of BC, the line IH be drawn parallel to the base AB, the point H will also be the middle of AD. For, because the figure AHIL is a parallelogram, as DHIK is likewise, their opposite sides being parallel, we have AH=IL, and DH=IK; but since the triangles BIL, CIK are equal, we already have IL=IK; therefore AH=DH.

It may be observed, that the line HI=AL is equal to $\frac{AB+CD}{2}$; hence the area of the trapezium may also be expressed by EF × HI: it is therefore equal to the altitude of the trapezium multiplied by the line which connects the middle points of its unparallel sides.

PROPOSITION VIII. THEOREM.

If a line AC *is divided into two parts* AB, BC, *the square described on the whole line* AC, *will include the square described on the part* AB, *the square described on the part* BC, *and twice the rectangle contained under* AB, BC; *a property which is thus expressed:*

$$AC^2 \text{ or } (AB+BC)^2=AB^2+BC^2+2\,AB \times BC.$$



Construct the square ACDE; take AF=AB; draw FG parallel to AC, and BH parallel to AE.

The square ACDE is divided into four parts; the first ABIF is the square described on AB, since we made AF=AB: the second IGDH is the square described on BC; for since we have ACA=E, and AB=AF, the difference AC—AB must be equal to the difference AE—AF, which gives BC=EF; but IG is equal to BC, and DG to EF, since the lines are parallel; therefore IGDH is equal to the square described on BC. And those two squares being taken away from the whole square, there remain the two rectangles BCGI, EFIH, each of which is measured by AB × BC: hence the proposition is true.

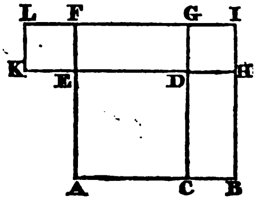
Scholium. This property is equivalent to the property demonstrated in algebra, for obtaining the square of a binomial, and expressed thus:

$$(a+b)^2 = a^2+2\,ab+b^2.$$

PROPOSITION IX. THEOREM.

If the line AC *is the difference of the two lines* AB, BC, *the square described on* AC *will be equivalent to the square of* AB, plus *the square of* BC, minus *twice the rectangle contained by* AB *and* BC ; *that is to say, we shall have*

$$AC^2 \text{ or } (AB - BC)^2 = AB^2 + BC^2 - 2\,AB \times BC.$$



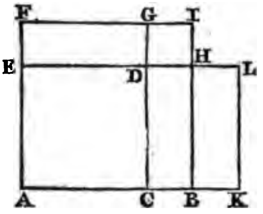
Construct the square ABIF; take AE= AC; draw CG parallel to BI, HK parallel to AB, and complete the square EFLK.

The two rectangles CBIG, GLKD are each measured by AB × BC; take them away from the whole figure ABILKEA, which is equivalent to AB^2+BC^2, there will evidently remain the square ACDE; hence our theorem is true.

Scholium. This proposition is equivalent to the algebraical formula, $(a-b)^2 = a^2 - 2ab + b^2$.

PROPOSITION X. THEOREM.

The rectangle contained by the sum and the difference of two lines, is equal to the difference of the squares of those lines: thus we have $(AB+BC) \times (AB - BC) = AB^2 - BC^2$.



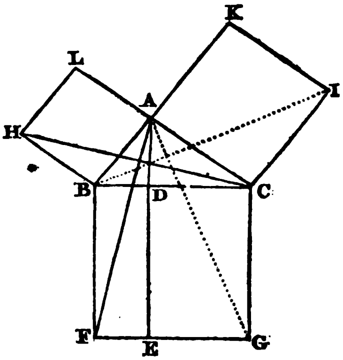
On AB and AC, construct the squares ABIF, ACDE; produce AB till the produced part BK is equal to BC; and complete the rectangle AKLE.

The base AK of the rectangle is the sum of the two lines AB, BC; its altitude AE is the difference of the same lines; therefore the rectangle AKLE is equal to (AB+BC) × (AB − BC). But this rectangle is composed of the two parts ABHE+BHLK; and the part BHLK is equal to the rectangle EDGF, because BH is equal to DE, and BK to EF; hence AKLE is equal to ABHE+EDGF. These two parts make up the square ABIF *minus* the square DHIG, which latter is the square described on BC: hence we have $(AB+BC) \times (AB - BC) = AB^2 - BC^2$.

Scholium. This proposition is equivalent to the algebraical formula, $(a+b)\,(a-b) = a^2 - b^2$.

PROPOSITION XI. THEOREM.

The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the two sides.



Let the triangle ABC be right-angled at A. Having formed squares on the three sides, let fall from A, on the hypotenuse, the perpendicular AD, which produce to E; and draw the diagonals AF, CH.

The angle ABF is made up of the angle ABC, together with the right angle CBF; the angle CBH is made up of the same angle ABC, together with the right angle ABH; hence the angle ABF is equal to HBC. But we have AB=BH, sides of the same square; and BF=BC, for the same reason: therefore the triangles ABF, HBC have two sides and the included angle in each equal; therefore (6. I.) they are themselves equal.

The triangle ABF (2. III.) is half of the rectangle BDEF (BE, for the sake of brevity), because they have the same base BF, and the same altitude BD. The triangle HBC is in like manner half of the square AH: for the angles BAC, BAL being both right, AC and AL form one and the same straight line parallel to HB; and consequently the triangle HBC, and the square AH, which have the common base BH, have also the common altitude AB, hence the triangle is half of the square.

The triangle ABF has already been proved equal to the triangle HBC; hence the rectangle BDEF, which is double of the triangle ABF, must be equivalent to the square AH, which is double of the triangle HBC. In the same manner it may be proved, that the rectangle CDEG is equivalent to the square AI. But the two rectangles BDEF, CDEG, taken together, make up the square BCGF: therefore the square BCGF, described on the hypotenuse, is equivalent to the sum of the squares ABHL, ACIK, described on the two other sides; in other words, $BC^2=AB^2+AC^2$.

Cor. 1. Hence the square of one of the sides of a right-angled triangle is equivalent to the square of the hypotenuse diminished by the square of the other side; which is thus expressed: $AB^2=BC^2-AC^2$.

Cor. 2. It has just been shewn that the square AH is equal to the rectangle BDEF: but by reason of the common altitude BF, the square BCGF is to the rectangle BDEF as the base BC is to the base BD; therefore we have

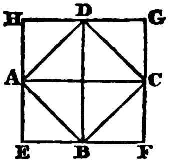
$$BC^2 : AB^2 :: BC : BD.$$

Hence *the square of the hypotenuse is, to the square of one of the sides about the right angle, as the hypotenuse is to the segment adjacent to that side.* The word *segment* here denotes that part of the hypotenuse, which is cut off by the perpendicular let fall from the right angle: thus BD is the segment adjacent to the side AB; and DC is the segment adjacent to the side AC. We might have, in like manner,

$$BC^2 : AC^2 :: BC : CD.$$

Cor. 3. The rectangles BDEF, DCGE, having likewise the same altitude, are to each other as their bases BD, CD. But these rectangles are equivalent to the squares AB^2, AC^2; therefore we have $AB^2 : AC^2 :: BD : DC$.

Hence *the squares of the two sides containing the right angle, are to each other as the segments of the hypotenuse, which lie adjacent to those sides.*



Cor. 4. Let ABCD be a square; and AC its diagonal: the triangle ABC being right-angled and isosceles, we shall have $AC^2 = AB^2 + BC^2 = 2AB^2$: hence *the square described on the diagonal* AC, *is double of the square described on the side* AB.

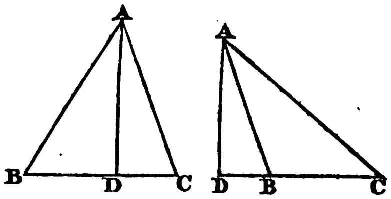
This property may be exhibited more plainly, by drawing parallels to BD, through the points A and C, and parallels to AC, through the points B and D. A new square EFGH will thus be formed, equal to the square of AC. Now EFGH evidently contains eight triangles each equal to ABE; and ABCD contains four such triangles: hence EFGH is double of ABCD.

Since we have $AC^2 : AB^2 :: 2 : 1$; by extracting the square roots, we shall have $AC : AB :: \sqrt{2} : 1$; hence *the diagonal of a square is incommensurable with its side*; a property which will be explained more fully in another place.

PROPOSITION XII. THEOREM.

In a triangle ABC, *if the angle* C *is acute, the square of the opposite side will be less than the sum of the squares of the sides which contain the angle* C ; *and* (AD *being drawn perpendicular to* BC), *the difference will be equal to twice the rectangle* $BD \times CD$: *so that we shall have*

$$AB^2 = AC^2 + BC^2 - 2\,BC \times CD.$$



There are two cases.

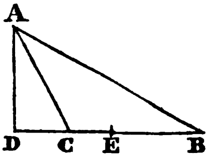
First. When the perpendicular falls within the triangle ABC, we have $BD = BC - CD$, and consequently (9. III.), $BD^2 = BC^2 + CD^2 - 2\,BC \times CD$. Adding AD^2 to each, and observing that the right-angled triangles ABD, ADC give $AD + BD^2 = AB^2$, and $AD^2 + CD^2 = AC^2$, we have $AB^2 = BC^2 + AC^2 - 2\,BC \times CD$.

Second. When the perpendicular AD falls without the triangle ABC, we have $BD = CD - BC$; and consequently (9. III.) $BD^2 = CD^2 + BC^2 - 2\,CD \times BC$. Adding AD^2 to both, we find, as before, $AB^2 = BC^2 + AC^2 - 2\,BC \times CD$.

PROPOSITION XIII. THEOREM.

In a triangle ABC, *if the angle* C *is obtuse, the square of the opposite side* AB, *will be greater than the sum of the squares of the sides, which contain the angle* C; *and* (AD *being drawn perpendicular to* BC), *the difference will be equal to twice the rectangle* $BC \times CD$: *so that we shall have*

$$AB^2 = AC^2 + BC^2 + 2BC \times CD.$$



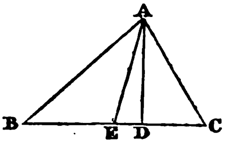
The perpendicular cannot fall within the triangle; for if it fell at any point such as E, the triangle ACE would have both the right angle E, and the obtuse angle C; which is impossible (27. I.) : hence the perpendicular falls without; and we have $BD = BC + CD$. From this (8. III.) there results $BD^2 = BC^2 + CD^2 + 2\,BC \times CD$. Adding AD^2 to both, and reducing the sums as in the last theorem, we find $AB^2 = BC^2 + AC^2 + 2BC \times CD$.

Scholium. The right-angled triangle is the only one in which the squares of two sides are together equal to the square of the third ; for if the angle contained by the two sides is acute, the sum of their squares will be greater than the square of the opposite side; if obtuse, it will be less.

PROPOSITION XIV. THEOREM.

In any triangle ABC, *if a straight line* AE *be drawn from the vertex to the middle of the base, we shall have*

$$AB^2+AC^2=2AE^2+2BE^2.$$



On BC, let fall the perpendicular AD.
The triangle AEC (12. III.) gives

$$AC^2=AE^2+EC^2-2EC\times ED$$

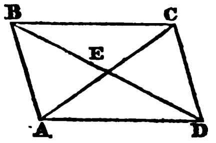
The triangle ABC (13. III.) gives

$$AB^2=AE^2+EB^2+2EB\times ED$$

Hence, by adding the corresponding sides together, and observing that EB and EC are equal, we have

$$AB^2+AC^2=2AE^2+2EB^2$$

Cor. Hence, *in every parallelogram the squares of the sides are together equal to the squares of the diagonals.*



For the diagonals AC, BD bisect each other (32. 1.); consequently the triangle ABC gives

$$AB^2+BC^2=2AE^2+2BE^2$$

The triangle ADC gives, in like manner,

$$AD^2+DC^2=2AE^2+2DE^2$$

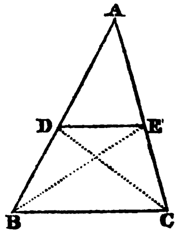
Adding the corresponding members together, and observing that BE and DE are equal, we shall have

$$AB^2+AD^2+DC^2+BC^2=4AE^2+4DE^2.$$

But $4AE^2$ is the square of 2AE, or of AC; $4DE^2$ is the square of BD: hence the squares of the sides are together equal to the squares of the diagonals.

PROPOSITION XV. THEOREM.

The line DE, *drawn parallel to the base of the triangle* ABC, *divides the sides* AB, BC *proportionally; so that we have* AD : DB : : AE : EC.



Join BE and DC. The two triangles BDE, DEC having the same base DE, and the same altitude, since both their vertices lie in a line parallel to the base, are equivalent (2. III.).

The triangles ADE, BDE, whose common vertex is E, have the same altitude, and (6. 3.) are to each other as their bases: hence we have

ADE : BDE : : AD : DB.

The triangles ADE, DEC, whose common

vertex is D, have also the same altitude, and are to each other as their bases; hence

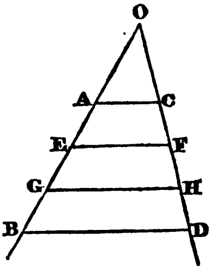
ADE : DEC : : AE : EC.

But the triangles BDE, DEC are equal; and therefore, since those proportions have a common ratio, we obtain

AD : DB : : AE : EC.

Cor. 1. Hence, *componendo,* we have AD + DB : AD : : AE+EC : AE, or AB : AD : : AC : AE; and also AB : BD : : AC : CE.

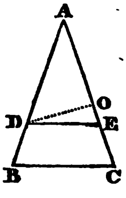
Cor. 2. If between two straight lines AB, CD, any number of parallels AC, EF, GH, BD, &c. be drawn, those straight lines will be cut proportionally, and we shall have AE : CF : : EG : FH : : GB : HD.



For, let O be the point where AB and CD meet. In the triangle OEF, the line AC being drawn parallel to the base EF, we shall have OE : AE : : OF : CF, or OE: OF : : AE : CF. In the triangle OGH, we shall likewise have OE : EG : : OF : FH, or OE : OF : : EG : FH. And by reason of the common ratio OE : OF, those two proportions give AE : CF : : EG : FH. It may be proved in the same manner, that EG : FH : : GB : HD, and so on; hence the lines AB, CD are cut proportionally by the parallels AC, EF, GH, &c.

PROPOSITION XVI. THEOREM.

Conversely, if the sides AB, AC, *are cut proportionally by the line* DE, *so that we have* AD : DB : : AE : EC, *the line* DE *will be parallel to the base* BC.

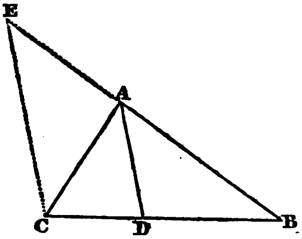


For if DE is not parallel to BC, suppose that DO is. Then, by the preceding theorem, we shall have AD : BD : : AO : OC. But by hypothesis, we have AD : DB : : AE : EC : hence we must have AO : OC : : AE : EC, or AO : AE : : OC : EC; an impossible result, since AO, the one antecedent, is less than its consequent AE, and OC, the other antecedent, is greater than its consequent EC. Hence the parallel to BC, drawn from the point D, cannot differ from DE; hence DE is that parallel.

Scholium. The same conclusion would be true, if the proportion AB : AD : : AC : AE were the proposed one. For this proportion would give us AB—AD : AD : : AC—AE : AE, or BD : AD : : CE : AE.

PROPOSITION XVII. THEOREM.

The line AD, *which bisects the angle* BAC *of a triangle, divides the base* BC *into two segments* BD, DC *proportional to the adjacent sides,* AB, AC; *so that we have* BD : DC : : AB : AC.



Through the point C, draw CE parallel to AD till it meet BA produced.

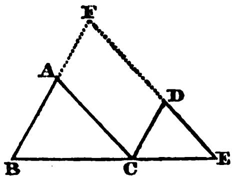
In the triangle BCE, the line AD is parallel to the base CE; hence (15. III.) we have the proportion,

BD : DC : : AB : AE.

But the triangle ACE is isosceles: for since AD, CE are parallel, we have the angle ACE = DAC, and the angle AEC = BAD (23. I.); and by hypothesis DAC=BAD; hence the angle ACE=AEC, and consequently (13. I.) AE=AC. In place of AE in the above proportion, substitute AC, and we shall have BD : DC : : AB : AC.

PROPOSITION XVIII. THEOREM.

Two equiangular triangles have their homologous sides proportional, and are similar.



Let ABC, CDE be two triangles which have their angles equal each to each, namely, BAC=CDE, ABC=DCE and ACB =DEC; then the homologous sides, or the sides adjacent to the equal angles, will be proportional, so that we shall have BC : CE : : AB : CD : : AC : DE.

Place the homologous sides BC, CE in the same straight line; and produce the sides BA, ED till they meet in F.

Since BCE is a straight line, and the angle BCA is equal to CED, it follows (23. I.) that AC is parallel to DE. In like manner, since the angle ABC is equal to DCE, the line AB is parallel to DC. Hence the figure ACDF is a parallelogram.

In the triangle BFE, the line AC is parallel to the base FE; hence (15. III.) we have BC : CE : : BA : AF; or, putting CD in the place of its equal AF,

BC : CE : : BA : CD.

In the same triangle BEF, if BF be considered as the base, CD is parallel to it; and we have the proportion BC : CE :: FD : DE; or putting AC in the place of its equal FD,

BC : CE :: AC : DE.

And finally, since both those proportions contain the same ratio BC : CE, we have

AC : DE :: BA : CD.

Thus the equiangular triangles BAC, CDE have their homologous sides proportional. But according to the second Definition, two figures are similar when they have their angles respectively equal, and their homologous sides proportional; consequently the equiangular triangles BAC, CDE, are two similar figures.

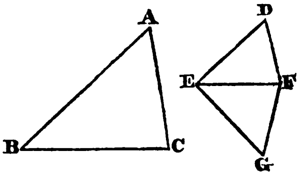
Cor. For the similarity of two triangles it is enough that they have two angles equal each to each; since the third will also be equal in both, and the two triangles will be equiangular.

Scholium. Observe that, in similar triangles, the homologous sides are opposite to the equal angles; thus the angle ACB being equal to DEC, the side AB is homologous to DC; in like manner, AC and DE are homologous, because opposite to the equal angles ABC, DCE. When the homologous sides are determined, it is easy to form the proportions:

AB : DC :: AC : DE :: BC : CE.

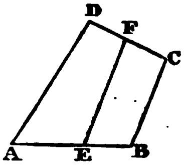
PROPOSITION XIX. THEOREM.

Two triangles, which have their homologous sides proportional, are equiangular and similar.



Suppose we have BC : EF :: AB : DE :: AC : DF; then will the triangles ABC, DEF have their angles equal, namely, A=D, B=E, C=F.

At the point E, make the angle FEG=B, and at F, the angle EFG=C; the third G will be equal to the third A, and the two triangles ABC, EFG will be equiangular. Therefore, by the last theorem, we shall have BC : EF :: AB : EG; but by hypothesis, we have BC : EF :: AB : DE; hence EG=DE. By the same theorem, we shall also have BC : EF :: AC : FG; and by hypothesis, we have BC : EF :: AC : DF; hence FG=DF. Hence (11. I.) the triangles EGF, DEF, having their three sides respectively equal, are themselves equal. But by construction, the triangles EGF and ABC are equiangular: hence DEF and ABC are also equiangular and similar.

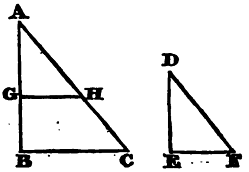


Scholium 1. By the last two propositions, it appears that in triangles, equality among the angles is a consequence of proportionality among the sides, and conversely; so that one of those conditions sufficiently determines the similarity of two triangles. The case is different with regard to figures of more than three sides: even in quadrilaterals, the proportion between the sides may be altered without altering the angles, or the angles be altered without altering the proportion between the sides; and thus proportionality among the sides cannot be a consequence of equality among the angles of two quadrilaterals, or *vice versa.* It is evident, for example, that by drawing EF parallel to BC, the angles of the quadrilateral AEFD, are made equal to those of ABCD, though the proportion between the sides is different; and, in like manner, without changing the four sides AB, BC, CD, AD, we can make the point B approach D or recede from it, which will change the angles.

Scholium 2. The two preceding propositions, which in strictness form but one, together with that relating to the square of the hypotenuse, are the most important and fertile in results of any in geometry: they are almost sufficient of themselves for every application to subsequent reasoning, and for solving every problem. The reason is, that all figures may be divided into triangles, and any triangle into two right-angled triangles. Thus the general properties of triangles include, by implication, those of all figures.

PROPOSITION XX. THEOREM.

Two triangles which have an equal angle included between proportional sides, are similar.

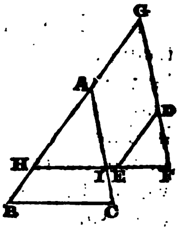


Let the angles A and D be equal; if we have AB : DE :: AC : DF, the triangle ABC is similar to DEF.

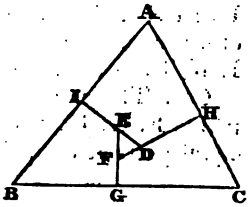
Take AG=DE, and draw GH parallel to BC. The angle AGH (23. I.) will be equal to the angle ABC; and the triangles AGH, ABC will be equiangular: hence we shall have AB : AG :: AC : AH. But by hypothesis, we have AB : DE :: AC : DF; and, by construction, AG=DE: hence AH=DF. The two triangles AGH, DEF have an equal angle included between equal sides; therefore they are equal; but the triangle AGH is similar to ABC; therefore DEF is also similar to ABC.

PROPOSITION XXI. THEOREM.

Two triangles, which have their homologous sides parallel, or perpendicular to each other, are similar.



First. If the side AB is parallel to DE, and BC to EF, the angle ABC (26. I.) will be equal to DEF; if AC is parallel to DF, the angle ACB will be equal to DFE, and also BAC to EDF; hence the triangles ABC, DEF are equiangular; hence they are similar.



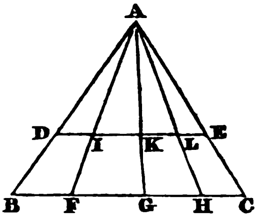
Secondly. If the side DE is perpendicular to AB, and the side DF to AC, the two angles I and H of the quadrilateral AIDH will be right; and since (28. I.) all the four angles are together equal to four right angles, the remaining two IAH, IDH will be together equal to two right angles. But the two angles EDF, IDH are also equal to two right angles: hence the angle EDF is equal to IAH or BAC. In like manner, if the third side EF is perpendicular to the third BC, it may be shewn that the angle DFE is equal to C, and DEF to B: hence the triangles ABC, DEF, which have the sides of the one perpendicular to the corresponding sides of the other, are equiangular and similar.

Scholium. In the case of the sides being parallel, the homologous sides are the parallel ones: in the case of their being perpendicular, the homologous sides are the perpendicular ones. Thus in the latter case DE is homologous with AB, DF with AC, and EF with BC.

The case of the perpendicular sides might present a relative position of the two triangles different from that exhibited in the diagram: but the equality of the respective angles might still be demonstrated, either by means of quadrilaterals like AIDH having two right angles, or by the comparison of two triangles having two of their angles vertical, and a right angle in each. Besides, we might always conceive a triangle DEF to be constructed within the triangle ABC, and such that its sides should be parallel to those of the triangle compared with ABC; and then the demonstration given in the text would apply.

PROPOSITION XXII. THEOREM.

The lines AF, AG, *&c. drawn anyhow through the vertex of a triangle, will divide* BC *the base, and* DE *parallel to the base, in the same proportion; so that we shall have* DI : BF :: IK : FG :: KL : GH, &c.



For since DI is parallel to BF, the triangles ADI and ABF are equiangular; and we have DI : BF :: AI : AF; and since IK is parallel to FG, we have in like manner AI : AF :: IK : FG; hence, the ratio AI : AF being common, we shall have DI : BF :: IK : FG. In the same manner we shall find IK : FG :: KL : GH; and so with the other segments: hence the line DE is divided at the points I, K, L, as the base BC at the points F, G, H.

Cor. Therefore if BC were divided into equal parts at the points F, G, H, the parallel DE would also be divided into equal parts at the points I, K, L.

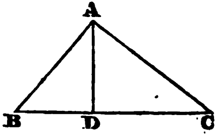
PROPOSITION XXIII. THEOREM.

If from the right angle A *of a right-angled triangle, the perpendicular* AD *be let fall on the hypotenuse; then,*

First, *The two partial triangles* ABD, ADC, *will be similar to each other and to the whole triangle* ABC;

Secondly, *Either side* AB *or* AC *will be a mean proportional between the hypotenuse* BC *and the adjacent segment* BD *or* DC; *and,*

Thirdly, *The perpendicular* AD *will be a mean proportional between the two segments* BD *and* DC.



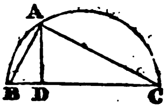
First. The triangles BAD and BAC have the common angle B, the right angle BDA=BAC, and therefore the third angle BAD of the one equal to the third C of the other: hence those two triangles are equiangular and similar. In the same manner it may be shewn that the triangles DAC and BAC are similar; hence all the three triangles are similar and equiangular.

Secondly. The triangles BAD, BAC being similar, their homologous sides are proportional. But BD in the little triangle and BA in the large one are homologous, because they lie opposite the equal angles BAD, BCA; the hypotenuse BA of the little triangle is homologous with the hypotenuse BC of the large triangle: hence the proportion BD : BA : : BA : BC. By the same reasoning, we should find DC : AC : : AC : BC; hence each of the sides AB, AC is a mean proportional between the hypotenuse and the segment adjacent to that side.

Thirdly. Since the triangles ABD, ADC are similar, by comparing their homologous sides, we have BD: AD: : AD : DC; hence, the perpendicular AD is a mean proportional between the segments DB, DC of the hypotenuse.

Scholium. Since BD : AB : : AB : BC, the product of the extremes will be equal to that of the means, or $AB^2=BD.BC$. For the same reason we have $AC^2=DC.BC$; therefore $AB^2+AC^2=BD.BC+DC.BC=(BD+DC).BC=BC.BC=BC^2$; or the square described on the hypotenuse BC is equal to the squares described on the two sides AB, AC. Thus we again arrive at the property of the square of the hypotenuse, by a path very different from that which formerly conducted us to it; and thus it appears that, in strict speaking, the property of the square of the hypotenuse is a consequence of the more general property, that the sides of equiangular triangles are proportional. Thus the fundamental propositions of geometry are reduced, as it were, to this single one, that equiangular triangles have their homologous sides proportional.

It happens frequently, as in this instance, that by deducing consequences from one or more propositions, we are led back to some proposition already proved. In fact, the chief characteristic of geometrical theorems, and one indubitable proof of their certainty is, that, however we combine them together, provided only our reasoning be correct, the results we obtain are always perfectly accurate. The case would be different, if any proposition were false or only approximately true; it would frequently happen that on combining the propositions together, the error would increase and become perceptible. Examples of this are to be seen in all the demonstrations, in which the *reductio ad absurdum* is employed. In such demonstrations, where the object is to shew that two quantities are equal, we proceed by shewing that if there existed the smallest inequality between the quantities, a train of accurate reasoning would lead us to a manifest and palpable absurdity; from which we are forced to conclude that the two quantities are equal.

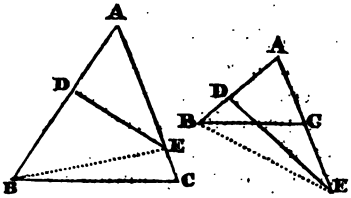


Cor. If from a point A, in the circumference of a circle, two chords AB, AC be drawn to the extremities of a diameter BC, the triangle BAC (Prop. 18. II.) will be right-angled at A; hence, first, *the perpendicular* AD *is a mean proportional between the two segments* BD, DC, *of the diameter*, or what amounts to the same, $AD^2=BD.DC$.

Hence also, in the second place, *the chord* AB *is a mean proportional between the diameter* BC *and the adjacent segment* BD, or what amounts to the same, $AB^2=BD.BC$. In like manner, we have $AC^2=CD.BC$; hence $AB^2:AC^2::BD:DC$; and comparing AB^2 and AC^2 to BC^2, we have $AB^2:BC^2::BD:BC$, and $AC^2:BC^2::DC:BC$. Those proportions between the squares of the sides compared with each other, or with the square of the hypotenuse, have already been given in the third and fourth corollaries of Prop. 11.

PROPOSITION XXIV. THEOREM.

Two triangles, having an equal angle, are to each other as the rectangles of the sides which contain that angle. Thus the triangle ABC *is to the triangle* ADE *as the rectangle* AB.AC *is to the rectangle* AD.AE.



Draw BE. The triangles ABE, ADE, having the common vertex E, have the same altitude, and consequently (Prop. 6.) are to each other as their bases: that is,

ABE : ADE : : AB : AD.

In like manner,

ABC : ABE : : AC : AE.

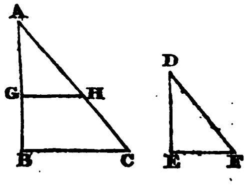
Multiply together the corresponding terms of those proportions, omitting the common term ABE; we have

ABC : ADE : : AB.AC : AD.AE.

Cor. Hence the two triangles would be equivalent, if the rectangle AB.AC were equal to the rectangle AD.AE, or if we had AB : AD : : AE : AC; which would happen if DC were parallel to BE.

PROPOSITION XXV. THEOREM.

Two similar triangles are to each other as the squares of their homologous sides.



Let the angle A be equal to D, and the angle B=E. Then, first, by reason of the equal angles A and D, according to the last proposition, we shall have

ABC : DEF : : AB.AC : DE.DF.

Also, because the triangles are similar,

AB : DE :: AC : DF.

And multiplying the terms of this proportion by the corresponding terms of the identical proportion,

AC : DF : : AC : DF,

there will result

$$AB.AC : DE.DF :: AC^2 : DF^2.$$

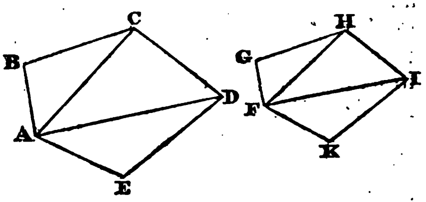
Consequently,

$$ABC : DEF :: AC^2 : DF^2.$$

Therefore two similar triangles ABC, DEF are to each other as the squares of the homologous sides AC, DF, or as the squares of any other two homologous sides.

PROPOSITION XXVI. THEOREM.

Two similar polygons are composed of the same number of triangles similar each to each, and similarly situated.

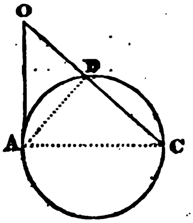


From any angle A, in the polygon ABCDE, draw diagonals AC, AD to the other angles. From the corresponding angle F, in the other polygon FGHIK, draw diagonals FH, FI to the other angles.

These polygons being similar, the angles ABC, FGH, which are homologous, must be equal (*Def.* 2.), and the sides AB, BC must also be proportional to FG, GH, that is, AB : FG : : BC : GH. Wherefore the triangles ABC, FGH have each an equal angle, contained between proportional sides; hence they are similar; hence the angle BCA is equal to GHF. Take away these equal angles from the equal angles BCD, GHI; there remains ACD=FHI. But since the triangles ABC, FGH are similar, we have AC : FH : : BC : GH; and (*Def.* 2.) since the polygons are similar, BC : GH : : CD : HI; hence AC : FH : : CD : HI. But the angle ACD, we already know, is equal to FHI; hence

PROPOSITION XXX. THEOREM.

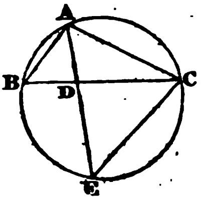
If from the same point O *without a circle, a tangent* OA *and a secant* OC *be drawn, the tangent will be a mean proportional between the secant and its external segment; that is, we shall have* OC : OA : : OA : OD; *or, which is the same thing,* $OA^2=OC.OD$.



For, joining AD and AC, the triangles OAD, OAC have the angle O common; also the angle OAD, formed by a tangent and a chord, has for its measure (19. II.) half of the arc AD; and the angle C has the same measure: hence the angle OAD=C; hence the two triangles are similar, and we have the proportion, OC : OA : : OA : OD, which gives $OA^2=OC.OD$.

PROPOSITION XXXI. THEOREM.*

In a triangle ABC, *if the angle* A *is bisected by the line* AD, *the rectangle of the sides* AB, AC *will be equal to the rectangle of the segments* BD, DC, *together with the square of* AD, *the bisecting line.*



Describe a circle through the three points A, B, C; produce AD till it meets the circumference, and join CE.

The triangle BAD is similar to the triangle EAC; for by hypothesis the angle BAD=EAC; also the angle B=E, since they both have for measure half of the arc AC; hence these triangles are similar, and the homologous sides give the proportion, BA : AE : : AD : AC; hence BA.AC=DE.AD; but AE=AD+DE, and multiplying each of these equals by AD, we have $AE.AD=AD^2+AD.DE$; now AD.DE=BD.DC (28. III.); hence finally,

$$BA.AC=AD^2+BD.DC.$$

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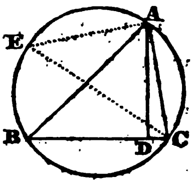
[*] This and the three succeeding propositions are not immediately connected with the chain of geometrical investigation. They may be omitted or not, as the reader chooses.—Ed.

PROPOSITION XXXII. THEOREM.

In every triangle ABC, *the rectangle of the two sides* AB, AC *is equal to the rectangle contained by the diameter* CE *of the circumscribed circle, and the perpendicular* AD *let fall upon the third side* BC.

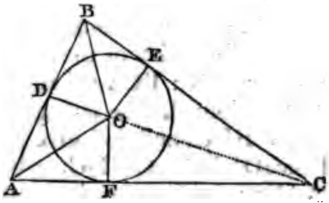


For, joining AE, the triangles ABD, AEC are right-angled, the one at D, the other at A; also the angle B=E; these triangles are therefore similar, and they give the proportion, AB : CE :: AD : AC; and hence AB.AC= CE.AD.

Cor. If these equal quantities be multiplied by the same quantity BC there will result AB.AC.BC=CE.AD.BC; now AD.BC is double of the surface of the triangle (6. III.); therefore *the product of the three sides of a triangle is equal to its surface multiplied by twice the diameter of the circumscribed circle.*

The product of three lines is sometimes called a *solid*, for a reason that shall be seen afterwards. Its value is easily conceived, by imagining that the lines are reduced into numbers, and multiplying these numbers together.

Scholium. It may also be demonstrated, that the surface of a triangle is equal to its perimeter multiplied by half the radius of the inscribed circle.

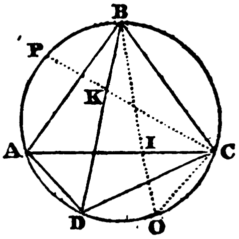


For the triangles AOB, BOC, AOC, which have a common vertex at O, have for their common altitude the radius of the inscribed circle; hence the sum of these triangles will be equal to the sum of the bases AB, BC, AC, multiplied by half the radius OD; hence the surface of the triangle ABC is equal to the perimeter multiplied by half the radius of the inscribed circle.

PROPOSITION XXXIII. THEOREM.

In every inscribed quadrilateral ABCD, *the rectangle of the two diagonals* AC, BD *is equal to the sum of the rectangles of the opposite sides; so that we have*

AC.BD=AB.CD+AD.BC.



Take the arc CO=AD, and draw BO meeting the diagonal AC in I.

The angle ABD=CBI, since the one has for its measure half of the arc AD, and the other half of CO equal to AD; the angle ADB=BCI, because they are both inscribed in the same segment AOB; hence the triangle ABD is similar to the triangle IBC, and we have the proportion AD : CI : : BD : BC; hence AD.BC=CI.BD. Again, the triangle ABI is similar to the triangle BDC; for the arc AD being equal to CO, if OD be added to each of them, we shall have the arc AO=DC; hence the angle ABI is equal to DBC; also the angle BAI to BDC, because they are inscribed in the same segment; hence the triangles ABI, DBC are similar, and the homologous sides give the proportion, AB : BD : : AI : CD; hence AB.CD=AI.BD.

Adding the two results obtained, and observing that AI.BD +CI.BD=(AI+CI).BD=AC.BD, we shall have AD.BC+ AB.CD=AC.BD.

Scholium. Another theorem concerning the inscribed quadrilateral may be demonstrated in the same manner:

The similarity of the triangles ABD and BIC gives the proportion BD : BC : : AB : BI; hence BI.BD=BC.AB. If CO be joined, the triangle ICO, similar to ABI, will be similar to BDC, and will give the proportion BD : CO : : DC : OI; hence OI.BD=CO.DC, or, because CO=AD, OI.BD=AD.DC. Adding the two results, and observing that BI.BD+OI.BD is the same as BO.BD, we shall have BO.BD=AB.BC+AD.DC.

If BP had been taken equal to AD, and CKP been drawn, a similar train of reasoning would have given us

CP.CA=AB.AD+BC.CD.

But the arc BP being equal to CO, if BC be added to each of them, it will follow that CBP=BCO; the chord CP is therefore equal to the chord BO, and consequently BO.BD and CP.CA are to each other as BD is to CA; hence,

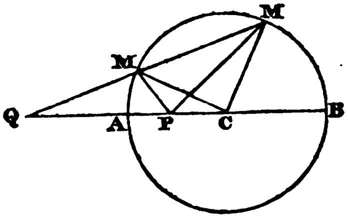
BD : CA : : AB.BC+AD.DC : AD.AB+BC.CD.

Therefore *the two diagonals of an inscribed quadrilateral are to each other, as the sums of the rectangles under the sides which meet at their extremities.*

These two theorems may serve to find the diagonals when the sides are given.

PROPOSITION XXXIV. THEOREM.

Let P *be a given point within a circle upon the radius* AC, *and let a point* Q *be taken externally upon the same radius produced, so that* CP : CA : : CA : CQ; *if from any point* M *of the circumference straight lines* MP, MQ *be drawn to the two points* P *and* Q, *these straight lines will every where have the same ratio, or* MP : MQ : : AP : AQ.

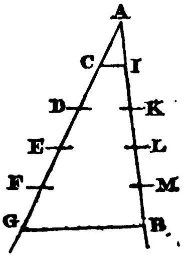


For by hypothesis, CP : CA : : CA : CQ; or substituting CM for CA, CP : CM : : CM : CQ; hence the triangles CPM, CQM, have each an equal angle O contained by proportional sides; hence they are similar; and hence the third side MP is to the third side MQ, as CP is to CM or CA. But (*dividendo*), the proportion CP : CA : : CA : CQ gives CP : CA : : CA—CP : CQ—CA, or CP : CA : : AP : AQ; therefore MP : MQ : : AP : AQ.

PROBLEMS RELATING TO THE THIRD BOOK.

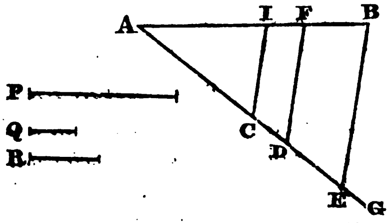
PROBLEM I.

To divide a given straight line into any number of equal parts, or into parts proportional to given lines.



First. Let it be proposed to divide the line AB into five equal parts. Through the extremity A, draw the indefinite straight line AG; and taking AC of any magnitude, apply it five times upon AG; join the last point of division G and the extremity B, by the straight line GB; then draw CI parallel to GB: AI will be the fifth part of the line AB; and thus, by applying AI five times upon AB, the line AB will be divided into five equal parts.

For since CI is parallel to GB, the sides AG, AB (15. III.) are cut proportionally in C and I. But AC is the fifth part of AG, hence AI is the fifth part of AB.

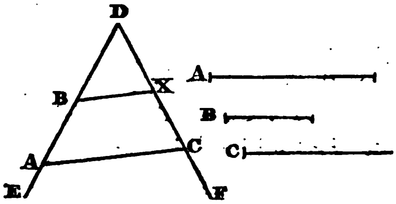


Secondly. Let it be proposed to divide the line AB into parts proportional to the given lines P, Q, R. Through A, draw the indefinite line AG; make AC=P, CD=Q, DE=R; join the extremities E and B; and through the points C, D draw CI, DF parallel to EB; the line AB will be divided into parts AI, IF, FB proportional to the given lines P, Q, R.

For, by reason of the parallels CI, DF, EB, the parts AI, IF, FB are proportional to the parts AC, CD, DE; and by construction, these are equal to the given lines P, Q, R.

PROBLEM II.

To find a fourth proportional to three given lines A, B, C.

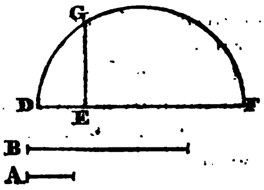


Draw the two indefinite lines DE, DF, forming any angle with each other. Upon DE take DA=A, and DB=B; upon DF take DC=G; join AC; and through the point B, draw BX parallel to AC; DX will be the fourth proportional required: for since BX is parallel to AC, we have the proportion DA : DB : : DC : DX; now the three first terms of that proportion are equal to the three given lines; consequently DX is the fourth proportional required.

Cor. A third proportional to two given lines A, B, may be found in the same manner, for it will be the same as a fourth proportional to the three lines A, B, B.

PROBLEM III.

To find a mean proportional between two given lines A *and* B.

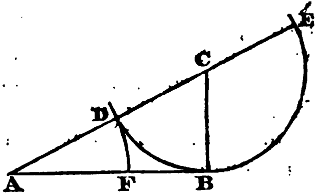


Upon the indefinite line DF, take DE=A, and EF=B; upon the whole line DF, as a diameter, describe the semicircle DGF; at the point E, erect upon the diameter the perpendicular EG meeting the circumference in G; EG will be the mean proportional required.

For the perpendicular EG, let fall from a point in the circumference upon the diameter, is a mean proportional between DE, DF, the two segments of the diameter (23. III.); and these segments are equal to the given lines A and B.

PROBLEM IV.

To divide the given line AB *into two parts, such that the greater part shall be a mean proportional between the whole line and the other part.*



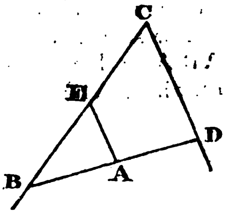
At the extremity B of the line AB, erect the perpendicular BC equal to the half of AB; from the point C as a centre, with the radius CB describe a semicircle; draw AC cutting the circumference in D; and take AF=AD: the line AB will be divided at the point F in the manner required; that is, we shall have AB : AF : : AF : FB.

For AB, being perpendicular to the radius at its extremity, is a tangent; and if AC be produced till it again meets the circumference in E, we shall have (30. III.) AE:AB::AB:AD; hence, *dividendo*, AE—AB : AB : : AB—AD : AD. But since the radius is the half of AB, the diameter DE is equal to AB, and consequently AE—AB=AD=AF; also, because AF=AD, we have AB—AD=FB; hence AF:AB::FB:AD or AF; whence, *invertendo*, AB : AF : : AF : FB.

Scholium. This sort of division of the line AB is called division in extreme and mean ratio: the use of it will be perceived in a future part of the work. It may farther be observed, that the secant AE is divided in extreme and mean ratio at the point D; for, since AB=DE, we have AE : DE : : DE : AD.

PROBLEM V.

Through a given point A, *in the given angle* BCD, *to draw the line* BD, *so that the segments* AB, AD, *comprehended between the point* A *and the two sides of the angle, shall be equal.*

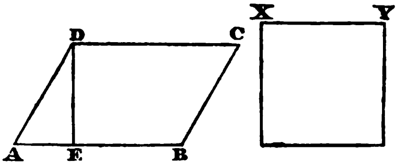


Through the point A draw AE parallel to CD, make BE=CE, and through the points B and A draw BAD; this will be the line required.

For, AE being parallel to CD, we have BE : EC : : BA : AD; but BE=EC; therefore BA=AD.

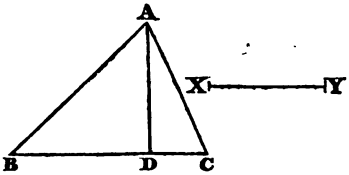
PROBLEM VI.

To describe a square that shall be equivalent to a given parallelogram or to a given triangle.



First. Let ABCD be the given parallelogram, AB its base, DE its altitude: between AB and DE find a mean proportional XY; then will the square constructed upon XY be equivalent to the parallelogram ABCD.

For by construction, AB : XY : : XY : DE; therefore XY^2=AB.DE; but AB.DE is the measure of the parallelogram, and XY^2 that of the square; consequently they are equivalent.

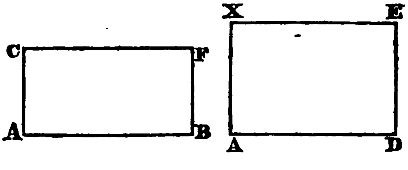


Secondly. Let ABC be the given triangle, BC its base, AD its altitude: find a mean proportional between BC and the half of AD, and let XY be that mean; the square constructed upon XY will be equivalent to the triangle ABC.

For since BC : XY : : XY : $\frac{1}{2}$ AD, it follows that XY^2= BC. $\frac{1}{2}$ AD; hence the square constructed upon XY is equivalent to the triangle ABC.

PROBLEM VII.

Upon the given line AD, *to describe a rectangle* ADEX *that shall be equivalent to a given rectangle* ABFC.

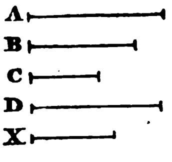


Find a fourth proportional to the three lines AD, AB, AC, and let AX be that fourth proportional; a rectangle constructed with the lines AD and AX will be equivalent to the rectangle ABFC.

For since AD : AB : : AC : AX, it follows that AD.AX= AB.AC; hence the rectangle ADEX is equivalent to the rectangle ABFC.

PROBLEM VIII.

To find two lines which shall have the same ratio to each other, as the rectangle of the two given lines A *and* B *has to the rectangle of the two given lines* C *and* D.



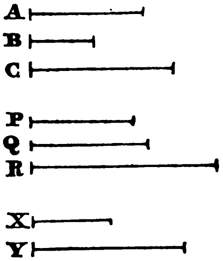
Let X be a fourth proportional to the three lines B, C, D; then will the two lines A and X have the same ratio to each other as the rectangles A.B and C.D.

For, since B : C : : D : X, it follows that C.D=B.X; hence A.B : C.D : : A.B : B.X : : A : X.

Cor. Hence to obtain the ratio of the squares constructed upon the given lines A and C, find a third proportional X to the lines A and C, so that A : C : : C : X; you will then have $A^2 : C^2 :: A : X$.

PROBLEM IX.

To find two lines that shall have the same ratio to each other as the product of the three given lines A, B, C *has to the product of the three given lines* P, Q, R.



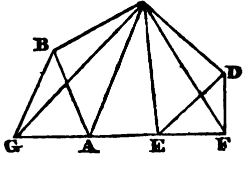
Find a fourth proportional X to the three given lines A, B, C: find also a fourth proportional Y to the three given lines P, Q, R. The two lines X, Y will be to each other as the products A.B.C, P.Q.R.

For since P : A : : B : X, it follows that A.B=P.X; and multiplying each of these equals by C, we have A.B.C=C.P.X. In like manner since C : Q : : R : Y, it follows that Q.R=C.Y; and multiplying each of these equals by P, we have P.Q.R=P.C.Y: hence the product A.B.C is to the product P.Q.R as C.P.X is to P.C.Y, or as X is to Y.

PROBLEM X.

To find a triangle that shall be equivalent to a given polygon.

Let ABCDE be the given polygon. Draw first the diagonal CE cutting off the triangle CDE; through the point D, draw DF parallel to CE, and meeting AE produced; join CF: the polygon ABCDE will be equivalent to the polygon ABCF, which has one side less than the original polygon.



For the triangles CDE, CFE have the base CE common; they have also the same altitude, since their vertices D, F, are situated in a line DF parallel to the base; these triangles are therefore equivalent. Add to each of them the figure ABCE, and there will result the polygon ABCDE equivalent to the polygon ABCF.

The angle B may in like manner be cut off, by substituting for the triangle ABC the equivalent triangle AGC, and thus the pentagon ABDE will be changed into an equivalent triangle GCF.

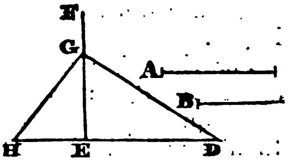
The same process may be applied to every other figure; for, by successively diminishing the number of its sides, one being retrenched at each step of the process, the equivalent triangle will at last be found.

Scholium. We have already seen that every triangle may be changed into an equivalent square; and thus a square may always be found equivalent to a given rectilineal figure, which operation is called *squaring* the rectilineal figure, or finding the *quadrature* of it.

The problem of *the quadrature of the circle* consists in finding a square equivalent to a circle whose diameter is given.

PROBLEM XI.

To find the side of a square which shall be equivalent to the sum or the difference of two given squares.



Let A and B be the sides of the given squares.

First. If it is required to find a square equivalent to the sum of these squares, draw the two indefinite lines ED, EF at right angles to each other; take ED=A, and EC=B; join DG: this will be the side of the square required.

For the triangle DEG being right-angled, the square constructed upon DG is equal to the sum of the squares upon ED and EG.

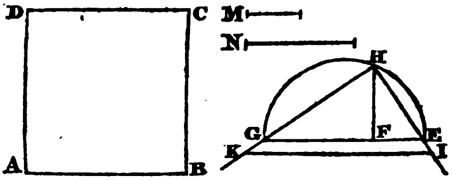
Secondly. If it is required to find a square equal to the difference of the given squares, form in the same manner the right angle FEH; take GE equal to the shorter of the sides A, and B; from the point G as a centre, with a radius GH, equal to the other side, describe an arc cutting EH in H: the square described upon EH will be equal to the difference of the squares described upon the lines A and B.

For the triangle GEH is right-angled, the hypotenuse GH=A, and the side GE=B; hence the square constructed upon EH, &c.

Scholium. A square may thus be found equal to the sum of any number of squares; for the construction which reduces two of them to one, will reduce three of them to two, and these two to one, and so of others. It would be the same, if any of the squares were to be subtracted from the sum of the others.

PROBLEM XII.

To construct a square which shall be to a given square ABCD *as the line* M *is to the line* N.

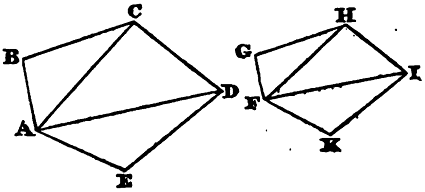


Upon the indefinite line EG, take EF=M, and FG=N; upon EG as a diameter describe a semicircle, and at the point F erect the perpendicular FH. From the point H, draw the chords HG, HE, which produce indefinitely: upon the first take HK equal to the side AB of the given square, and through the point K draw KI parallel to EG; HI will be the side of the square required.

For, by reason of the parallels KI, GE, we have HI : HK : : HE : HG ; hence $HI^2 : HK^2 :: HE^2 : HG^2$: but in the right-angled triangle EHG (23. III.) the square of HE is to the square of HG as the segment EF is to the segment FG, or as M is to N; hence $HI^2 : HK^2 :: M : N$. But HK=AB; therefore the square described upon HI is to the square described upon AB as M is to N.

PROBLEM XIII.

Upon the side FG, *homologous to* AB, *to describe a polygon similar to the given polygon* ABCDE.



In the given polygon, draw the diagonals AC, AD; at the point F make the angle GFH=BAC, and at the point G the angle FGH=ABC; the lines FG, GH will cut each other in H, and FGH will be a triangle similar to ABC. In the same manner upon FH, homologous to AC, construct the triangle FIH similar to ADC; and upon FI, homologous to AD, construct the triangle FIK similar to ADE. The polygon FGHIK will be similar to ABCDE, as required.

For, these two polygons are composed of the same number of triangles, which are similar and similarly situated (26. III.)

PROBLEM XIV.

Two similar figures being given, to construct a figure which shall be similar to one of them, and equal to their sum or their difference.

Let A and B be two homologous sides of the given figures. Find a square equal to the sum or to the difference of the squares described upon A and B; let X be the side of that square; then will X in the figure required, be the side which is homologous to the sides A and B in the given figures. The figure itself may then be constructed on X, by the last problem.

For, the similar figures are as the squares of their homologous sides; now the square of the side X is equal to the sum, or to the difference, of the squares described upon the homologous sides A and B; therefore the figure described upon the side X is equal to the sum, or to the difference, of the similar figures described upon the sides A and B.

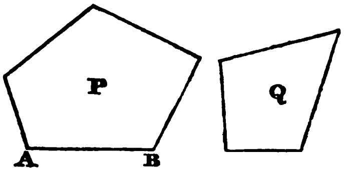
PROBLEM XV.

To construct a figure similar to a given one, and bearing to it the given ratio of M *to* N.

Let A be a side of the given figure, X the homologous side of the figure required. The square of X must be to the square of A as M is to N; hence X will be found by Problem 12.; and knowing X, the rest will be accomplished by Problem 13.

PROBLEM XVI.

To construct a figure similar to the figure P *and equivalent to the figure* Q.

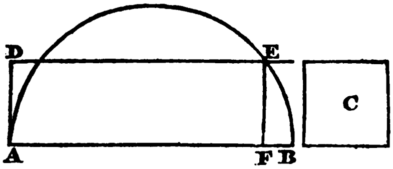


Find M the side of a square equivalent to the figure P, and N the side of a square equivalent to the figure Q. Let X be a fourth proportional to the three given lines M, N, AB; upon the side X, homologous to AB, describe a figure similar to the figure P; it will also be equivalent to the figure Q.

For, calling Y the figure described upon the side X, we have $P : Y :: AB^2 : X^2$; but by construction, $AB : X :: M : N$, or $AB^2 : X^2 :: M^2 : N^2$; hence $P : Y :: M^2 : N^2$. But by construction also, $M^2 = P$ and $N^2 = Q$; therefore $P : Y :: P : Q$; consequently $Y = Q$; hence the figure Y is similar to the figure P, and equivalent to the figure Q.

PROBLEM XVII.

To construct a rectangle equivalent to a given square C, *and having its adjacent sides together equal to a given line* AB.



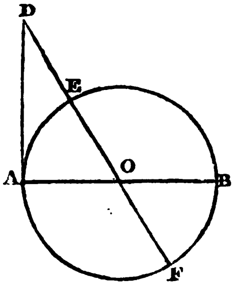
Upon AB as a diameter, describe a semicircle; draw the line DE parallel to the diameter, at a distance AD equal to the side of the given square C; from the point E, where the parallel cuts the circumference, draw EF perpendicular to the diameter; AF and FB will be the sides of the rectangle required.

For, their sum is equal to AB; and their rectangle AF.FB is equal to the square of EF, or to the square of AD; hence that rectangle is equivalent to the given square C.

Scholium. To render the problem possible, the distance AD must not exceed the radius; that is, the side of the square C must not exceed the half of the line AB.

PROBLEM XVIII.

To construct a rectangle that shall be equivalent to a given square C, *and the difference of whose adjacent sides shall be equal to a given line* AB.

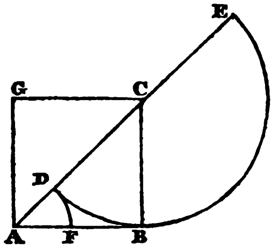


Upon the given line AB as a diameter, describe a semicircle; at the extremity of the diameter draw the tangent AD, equal to the side of the square C; through the point D and the centre O draw the secant DF: then will DE and DF be the adjacent sides of the rectangle required.

For, first, the difference of their sides is equal to the diameter EF or AB; secondly, the rectangle DE, DF is equal to AD^2 (30. III.); hence that rectangle is equivalent to the given square C.

PROBLEM XIX.

To find the common measure, if there is one, between the diagonal and the side of a square.



Let ABCG be any square whatever, and AC its diagonal.

We must first (Prob. 17. II) apply CB upon CA, as often as it may be contained there. For this purpose, let the semicircle DBE be described, from the centre C, with the radius CB. It is evident that CB is contained once in AC, with the remainder AD; the result of the first operation is therefore the quotient 1 with the remainder AD, which latter must now be compared with BC, or its equal AB.

We might here take AF=AD, and actually apply it upon AB; we should find it to be contained twice with a remainder: but as that remainder, and those which succeed it, continue diminishing, and would soon elude our comparisons by their minuteness, this would be but an imperfect mechanical method, from which no conclusion could be obtained to determine whether the lines AC, CB have or have not a common measure. There is a very simple way, however, of avoiding these decreasing lines, and obtaining the result, by operating only upon lines which remain always of the same magnitude:

The angle ABC being right, AB is a tangent, and AE a secant drawn from the same point; so that, (30. III.) AD : AB : : AB : AE. Hence in the second operation, when AD is compared with AB, the ratio of AB to AE may be taken instead of that of AD to AB; now AB, or its equal CD, is contained twice in AE, with the remainder AD; the result of the second operation is therefore the quotient 2 with the remainder AD, which must be compared with AB.

Thus the third operation again consists in comparing AD with AB, and may be reduced in the same manner to the comparison of AB or its equal CD with AE; from which there will again be obtained 2 for the quotient, and AD for the remainder.

Hence, it is evident, the process will never terminate; and therefore there is no common measure between the diagonal and the side of a square: a truth which was already known by arithmetic (since these two lines are to each other : : 2 : 1), but which acquires a greater degree of clearness by the geometrical investigation.

Scholium. The impossibility of finding numbers to express the exact ratio of the diagonal to the side of a square has now been proved; but an approximation may be made to it, as near as we please, by means of the continued fraction which is equal to that ratio. The first operation gave us 1 for a quotient; the second, and all the others to infinity, give two: thus the fraction in question is

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\text{ \&c. to infinity.}$$

If that fraction, for example, is computed to the fourth term inclusively, its value is found to be $1\frac{12}{29}$, or $\frac{41}{29}$; so that the approximate ratio of the diagonal to the side of a square, is : : 41 : 29. A closer approximation to the ratio might be found by computing a greater number of terms.

BOOK IV.

REGULAR POLYGONS, AND THE MEASUREMENT OF THE CIRCLE.

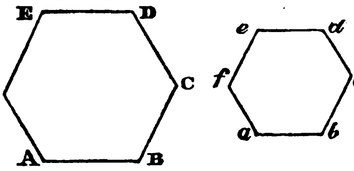
Definition.

A POLYGON, which is at once equilateral and equiangular, is called a *regular polygon.*

Regular polygons may have any number of sides: the equilateral triangle is one of three sides; the square is one of four.

PROPOSITION I. THEOREM.

Two regular polygons of the same number of sides are similar figures.



Suppose, for example, that ABCDEF, *abcdef,* are two regular hexagons. The sum of all the angles is the same in both figures, being in each equal to eight right angles (28. I.). The angle A is the sixth part of that sum; so is the angle *a*: hence the angles A and *a* are equal; and for the same reason, the angles B and *b*, the angles C and *c*, and so on.

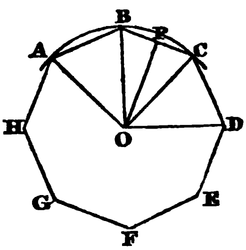
Again, since from the nature of the polygons, the sides AB, BC, CD, &c. are equal, and likewise the sides *ab, bc, cd,* &c.; it is plain that AB : *ab* : : BC : *bc* : : CD ; *cd,* &c.; hence the two figures in question have their angles equal, and their homologous sides proportional; hence (*Def.* 2. III.) they are similar.

Cor. The perimeters of two regular polygons of the same number of sides, are to each other as the homologous sides, and their surfaces as the squares of those sides (27. III.).

Scholium. The angle of a regular polygon is determined by the number of sides, as that of an equiangular polygon (28. I.)

PROPOSITION II. THEOREM.

Any regular polygon may be inscribed in a circle, and circumscribed about one.



Let ABCDE, &c. be a regular polygon: describe a circle through the three points A, B, C, the centre being O, and OP the perpendicular let fall from it to the middle point of BC: join AO and OD.

If the quadrilateral OPCD be placed above the quadrilateral OPBA, they will coincide; for the side OP is common: the angle OPC=OPB, being right; hence the side PC will apply to its equal PB, and the point C will fall on B: besides, from the nature of the polygon, the angle PCD=PBA; hence CD will take the direction BA; and since CD=BA, the point D will fall on A, and the two quadrilaterals will entirely coincide. The distance OD is therefore equal to AO; and consequently the circle which passes through the three points A, B, C, will also pass through the point D. By the same mode of reasoning, it might be shewn, that the circle which passes through the three points B, C, D, will also pass through the point E; and so of all the rest: hence the circle which passes through the points A, B, C, passes through the vertices of all the angles in the polygon, which is therefore inscribed in this circle.

Again, in reference to this circle, all the sides AB, BC, CD, &c. are equal chords; they are therefore (8. II.) equally distant from the centre: hence, if from the point O with the distance OP, a circle be described, it will touch the side BC, and all the other sides of the polygon, each in its middle point, and the circle will be inscribed in the polygon, or the polygon described about the circle.

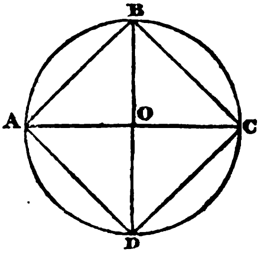
Scholium 1. The point O, the common centre of the inscribed and circumscribed circles, may also be regarded as the centre of the polygon; and upon this principle the angle AOB is called *the angle at the centre*, being formed by two radii drawn to the extremities of the same side AB.

Since all the chords AB, BC, CD, &c. are equal, all the angles at the centre must evidently be equal likewise; and therefore the value of each will be found by dividing four right angles by the number of the polygon's sides.

Scholium 2. To inscribe a regular polygon of a certain number of sides in a given circle, we have only to divide the circumference into as many equal parts as the polygon has sides: for the arcs being equal (see the diagram of Prop. 4.) the chords AB, BC, CD, &c. will also be equal; hence likewise the triangles ABO, BOC, COD must be equal, because they are equiangular; hence all the angles ABC, BCD, CDE, &c. will be equal; hence the figure ABCDE &c. will be a regular polygon.

PROPOSITION III. PROBLEM.

To inscribe a square in a given circle.

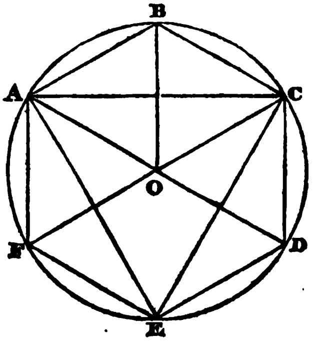


Draw two diameters AC, BD, cutting each other at right angles; join their extremities A, B, C, D: the figure ABCD will be a square. For the angles AOB, BOC, &c. being equal, the chords AB, BC, &c. are also equal; and the angles ABC, BCD, &c. being in semicircles, are right.

Scholium. Since the triangle is right-angled and isosceles, we have (11. III.) BC : BO :: $\sqrt{2}$: 1; hence *the side of the inscribed square is to the radius, as the square root of 2 is to unity.*

PROPOSITION IV. PROBLEM.

In a given circle, to inscribe a regular hexagon and an equilateral triangle.



Suppose the problem solved, and that AB is a side of the inscribed hexagon; the radii AO, OB being drawn, the triangle AOB will be equilateral.

For the angle AOB is the sixth part of four right angles; therefore, taking the right angle for unity, we shall have AOB$=\frac{4}{6}=\frac{2}{3}$: and the two other angles ABO, BAO, of the same triangle, are together equal to $2-\frac{2}{3}=\frac{4}{3}$; and being mutually equal, each of them must be equal to $\frac{2}{3}$; therefore the triangle ABO is equilateral; therefore the side of the inscribed hexagon is equal to the radius.

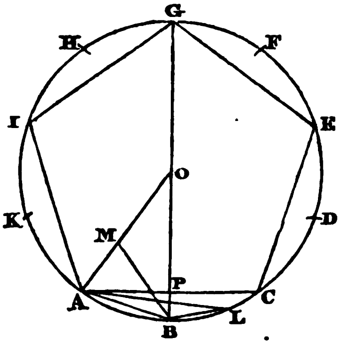
Hence to inscribe a regular hexagon in a given circle, the radius must be applied six times to the circumference; which will bring us round to the point we set out from.

And the hexagon ABCDEF being inscribed, the equilateral triangle ACE may be formed by joining the vertices of the alternate angles.

Scholium. The figure ABCO is a parallelogram, and even a rhombus, since AB=BC=CO=AO; hence (14. III.) the sum of the squares of the diagonals AC^2+BO^2 is equal to the sum of the squares of the sides, that is, to $4AB^2$, or $4BO^2$: and taking away BO from both, there will remain $AC^2=3\,BO^2$; hence $AC^2 : BO^2 :: 3 : 1$, or $AC : BO :: \sqrt{3} : 1$; hence *the side of the inscribed equilateral triangle is to the radius, as the square root of three is to unity.*

PROPOSITION V. PROBLEM.

In a given circle, to inscribe a regular decagon; then a pentagon, and a pentedecagon.



Divide the radius AO in extreme and mean ratio (Prob. 4 III.) at the point M; take the chord AB equal to OM the greater segment; AB will be the side of the regular decagon, and will require to be applied ten times to the circumference.

For, joining MB, we have by construction, AO : OM :: OM : AM; or, since AB=OM, AO : AB :: AB : AM; hence the triangles ABO, AMB have a common angle A, included between proportional sides; hence (20. III.) they are similar. Now the triangle OAB being isosceles, AMB must be isosceles also, and AB=BM; besides AB=OM; hence also MB=OM; hence the triangle BMO is isosceles.

Again, the angle AMB being exterior to the isosceles triangle BMO, is double of the interior angle O (27. I.): but the angle AMB=MAB; hence the triangle OAB is such, that each of the angles at its base, OAB or OBA, is double of O the angle at its vertex; hence the three angles of the triangle are together equal to five times the angle O, which consequently is the fifth part of the two right angles, or the tenth part of four hence the arc AB is the tenth part of the circumference, and the chord AB is the side of the regular decagon.

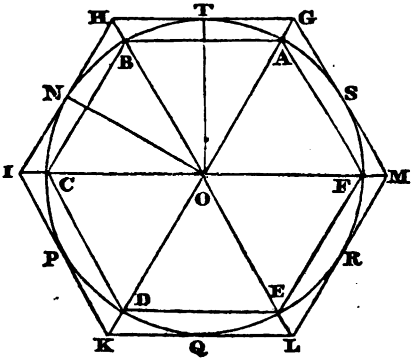
Cor. 1. By joining the alternate corners of the regular decagon, the pentagon ACEGI will be formed, also regular.

Cor. 2. AB being still the side of the decagon, let AL be the side of the hexagon; the arc BL will then, with reference to the whole circumference, be $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$; hence the chord BL will be the side of the pentedecagon or regular polygon of fifteen sides. It is evident, also, that the arc CL is the third of CB.

Scholium. Any regular polygon being inscribed, if the arcs subtended by its sides be severally bisected, the chords of those semi-arcs will form a new regular polygon of double the number of sides: thus it is plain, the square may enable us successively to inscribe regular polygons of 8, 16, 32, &c. sides. And in like manner, by means of the hexagon, regular polygons of 12, 24, 48, &c. sides may be inscribed; by means of the decagon, polygons of 20, 40, 80, &c. sides; by means of the pentedecagon, polygons of 30, 60, 120, &c. sides.*

PROPOSITION VI. PROBLEM.

A regular inscribed polygon ABCD &c. *being given, to circumscribe a similar polygon about the same circle.*



At T, the middle point of the arc AB, apply the tangent GH, which (10. II.) will be parallel to AB; do the same at the middle point of each of the arcs BC, CD, &c.; those tangents, by their intersections, will form the regular circumscribed polygon GHIK &c., similar to the inscribed one.

It is evident, in the first place, that the three points O, B, H, lie in the same straight line; for the right-angled triangles OTH, OHN, having the common hypotenuse OH, and the side OT=ON, must be equal, and consequently the angle TOH=HON, wherefore the line OH passes through the middle point B of the arc TN. For a like reason, the point I is in the prolongation of OC; and so with the rest. But since BH is parallel to AB, and HI to BC, the angle GHI=ABC (26. I.); in like manner, HIK=BCD; and so with all the rest: hence the angles of the circumscribed polygon are equal to those of the inscribed one. And farther, by reason of these same parallels, we have GH : AB : : OH : OB, and HI : BC : : OH : OB; therefore GH : AB : : HI : BC. But AB = BC, therefore GH = HI. For the same reason, HI=IK, &c.; hence the sides of the circumscribed polygon are all equal; hence this polygon is regular, and similar to the inscribed one.

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* It was long supposed, that, besides the polygons here mentioned, no other could be inscribed by the operations of elementary geometry, or what amounts to the same, by the resolution of equations of the first and second degree. But M. Gauss of Göttingen at length proved, in a work entitled *Disquisitiones Arithmeticae, Lipsiae,* 1801, that by the method in question, a regular polygon of 17 sides might be inscribed, and generally a regular polygon of 2^n+1 sides, provided 2^n+1 be a prime number.

Cor. 1. Reciprocally, if the circumscribed polygon GHIK &c. were given, and the inscribed one ABC &c. were required to be deduced from it, it would only be necessary to draw from the angles G, H, I, &c. of the given polygon, straight lines OG, OH, &c. meeting the circumference in the points A, B, C, &c.; then to join those points by the chords AB, BC, &c.; which would form the inscribed polygon. An easier solution of this problem would be simply to join the points of contact T, N, P, &c. by the chords TN, NP, &c. which likewise would form an inscribed polygon similar to the circumscribed one.

Cor. 2. Hence we may circumscribe about a circle any regular polygon, which can be inscribed within it; and conversely.

PROPOSITION VII. THEOREM.

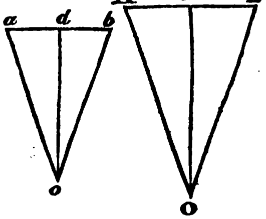
The area of a regular polygon is equal to its perimeter multiplied by half the radius of the inscribed circle.

Let the regular polygon be GHIK &c. (see the last figure), the triangle GOH will be measured by GH × ½OT; the triangle OHI by HI × ½ON: but ON=OT; hence the two triangles taken together will be measured by (GH+HI) × ½OT. And, by continuing the same operation for the other triangles, it will appear that the sum of them all, or the whole polygon, is measured by the sum of the bases GH, HI, IK, &c. or the perimeter of the polygon, multiplied into ½OT, or half the radius of the inscribed circle.

Scholium. The radius OT of the inscribed circle is nothing else than the perpendicular let fall from the centre to one of the sides: it is sometimes named the *apothem* of the polygon.

PROPOSITION VIII. THEOREM.

The perimeters of two regular polygons, having the same number of sides, are to each other as the radii of the circumscribed circles, and also as the radii of the inscribed circles; their surfaces are to each other as the squares of those radii.

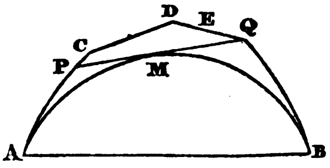


Let AB be a side of the one polygon, O the centre, and consequently OA the radius of the circumscribed circle, and OD, perpendicular to AB, the radius of the inscribed circle; let *ab*, in like manner, be a side of the other polygon, *o* its centre, *oa* and *od* the radii of the circumscribed and the inscribed circle. The perimeters of the two polygons are to each other as the sides AB and *ab*: but the angles A and *a* are equal, being each half of the angle of the polygon; so also are the angles B and *b*; hence the triangles ABO, *abo* are similar, as are likewise the right-angled triangles ADO, *ado*; hence AB : *ab* :: AO : *ao* :: DO : *do*; hence the perimeters of the polygons are to each other as the radii AO, *ao* of the circumscribed circles, and also as the radii DO, *do* of the inscribed circles.

The surfaces of those polygons are to each other as the squares of the homologous sides AB, *ab*; they are therefore likewise to each other as the squares of AO, *ao* the radii of the circumscribed circles, or as the squares of OD, *od* the radii of the inscribed circles.

PROPOSITION IX. LEMMA.

Any curve, or any polygonal line, which envelopes the convex line AMB *from one end to the other, is longer than* AMB *the enveloped line.*

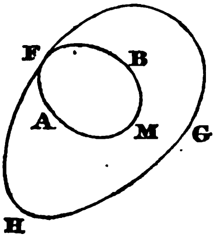


We have already said that by the term convex line, we understand a line, polygonal or curve, or partly curve and partly polygonal, such that a straight line cannot cut it in more than two points. If in the line AMB there were any sinuosities or re-entrant portions, it would cease to be convex, because a straight line might evidently cut it in more than two points. The arcs of a circle are essentially convex;

but the present proposition extends to any line which fulfils the required condition.

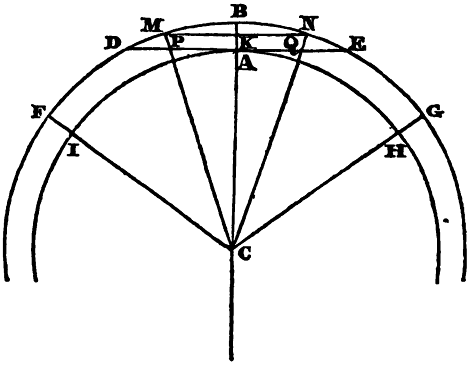
This being premised, if the line AMB is not shorter than any of those which envelope it, there will be found among the latter a line shorter than all the rest, which is shorter than AMB, or, at most, equal to it. Let ACDEB be this enveloping line: any where between those two lines draw the straight line PQ, not meeting, or at least only touching, the line AMB. The straight line PQ is shorter than PCDEQ; hence if, instead of the part PCDEQ, we substitute the straight line PQ, the enveloping line APQB will be shorter than APDQB. But, by hypothesis, this latter was shorter than any other; hence that hypothesis was false; hence all of the enveloping lines are longer than AMB.

Scholium. In the very same way, it might be shewn that AMB, a convex line returning into itself, is shorter than any line enveloping it on all sides, whether the enveloping line FHG touch AMB in one or more points, or surround without touching it.



PROPOSITION X. LEMMA.

Two concentric circles being given, a regular polygon may always be inscribed within the greater, the sides of which shall not meet the circumference of the less; and likewise, a regular polygon may always be described about the less, the sides of which shall not meet the circumference of the greater.



Let CA, CB be radii of the given circles. At the point A, apply the tangent DE, terminating in the greater circumference at D and E: inscribe within this greater circumference one of the regular polygons, which the methods already explained enable you to inscribe; next bisect the arcs subtended by its sides, and draw the chords of those half-arcs; a polygon will thus be found having twice as many sides. Continue the bisection, till an arc is obtained less than DBE. Let MBN be that arc, the middle point of it being supposed to lie at B: it is plain that the chord MN will be farther from the centre than DE; and that consequently the regular polygon, of which MN is a side, cannot meet the circumference, of which CA is the radius.

Now, the same construction remaining, join CM and CN, meeting the tangent DE in P and Q; PQ will be the side of a polygon described about the less circumference, similar to that polygon inscribed within the greater, of which the side is MN. And it is evident, that this circumscribed polygon having PQ for its side, can never meet the greater circumference, CP being less than CM.

Hence, by the same operation, a regular polygon may be inscribed within the greater circumference, and a similar one described about the less, both of which shall have their sides included between the two circumferences.

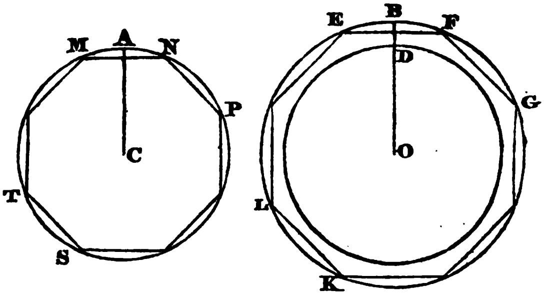
Scholium. If two concentric sectors FCG, ICH be given, a *portion of a regular polygon* may, in like manner, be inscribed in the greater, or circumscribed about the less, so that the perimeters of the two polygons shall be included between the two circumferences. For this purpose, it will be sufficient to divide the arc FBG successively into 2, 4, 8, 16, &c. equal parts, till a part smaller than DBE is obtained.

By the phrase, *portion of a regular polygon*, we here mean the figure terminated by a series of equal chords inscribed in the arc FG, from one of its extremities to the other. This portion has all the main properties of regular polygons; it has its angles equal, and its sides equal, it can be inscribed in a circle, or circumscribed about one: yet, properly speaking, it forms part of a regular polygon only in those cases where the arc subtended by one of its sides is an aliquot part of the circumference.

PROPOSITION XI. THEOREM.

The circumferences of circles are to each other as their radii, and their surfaces as the squares of those radii.

For the sake of brevity, let us designate the circumference whose radius is CA by *circ.* CA; we are to shew that *circ.* CA : *circ.* OB :: CA : OB.



If this proposition is not true, CA must be to OB as *circ.* CA is to a fourth term less or greater than *circ.* OB: suppose it less; and that, if possible, CA : OB :: *circ.* CA : *circ.* OD.

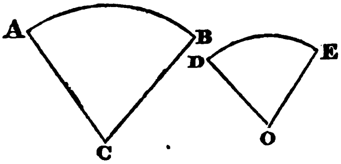
In the circle of which OB is the radius inscribe a regular polygon EFGKLE, such that the sides of it shall not meet the circumference of which OD is the radius (Prop. 10. IV.): inscribe a similar polygon, MNPFM, in the circle of which AC is the radius.

Then, since those polygons are similar, their perimeters MNPSM, EFGKE will be to each other (8. IV.) as CA, OB, the radii of the circumscribed circles, that is MNPSM : EFGKE : : CA : OB. But by hypothesis, CA : OB : : *circ.* CA: *circ.* OD; therefore MNPSM: EFGKE :: *circ.* CA : *circ.*OD; which proportion is false, because (9. IV.) the perimeter MNSPM is less than *circ.* CA, while on the contrary EFGKE is greater than *circ.* OD: therefore it is impossible that CA can be to OB as *circ.* CA is to a circumference less than *circ.* OB; or, in more general terms, it is impossible that one radius can be to another, as the circumference described with the former radius is to a circumference less than the one described with the latter radius.

Hence, too, we conclude it to be equally impossible that CA can be to OB as *circ.* CA is to a circumference greater than *circ.* OB : for if this were the case, by reversing the ratios, we should have OB to CA as a circumference greater than *circ.* OB is to *circ.* CA ; or, what amounts to the same thing, as *circ.* OB is to a circumference less than *circ.* CA ; and therefore one radius would be to another as the circumference described with the former radius is to a circumference less than the one described with the latter radius,—a conclusion just shewn to be erroneous.

And since the fourth term of this proportion CA : OB : : *circ.* CA : x can neither be greater nor less than *circ.* OB, it must be equal to *circ.* OB : consequently the circumferences of circles are to each other as their radii.

By the same construction, a similar train of reasoning would shew, that the surfaces of circles are to each other as the squares of their radii. We need not enter upon any farther details respecting this proposition, particularly as it forms a corollary of the next theorem.



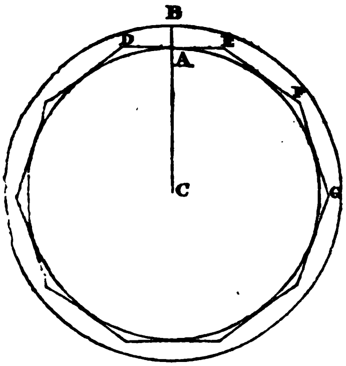
Cor. The similar arcs AB, DE are to each other as their radii AC, DO; and the similar sectors ACB, DOE are to each other as the squares of those radii.

For, since the arcs are similar, the angle C (Def. 3. III.) is equal to the angle O; but C is to four right angles (17. II.), as the arc AB is to the whole circumference described with the radius AC ; and O is to four right angles, as the arc DE is to the circumference described with the radius OD : hence the arcs AB, DE are to each other as the circumferences of which they form part; but these circumferences are to each other as their radii AC, DO; hence *arc.* AB : *arc.* DE : : AC : DO.

For a like reason, the sectors ACB, DOE are to each other as the whole circles; which again are as the squares of their radii; therefore *sect.* ACB ; *sect.* DOE : AC^2 : DO^2.

PROPOSITION XII. THEOREM.

The area of a circle is equal to the product of its circumference by half the radius



Let us designate the surface of the circle whose radius is CA by *surf.* CA; we shall have *surf.* CA$=\frac{1}{2}$CA × *circ.* CA.

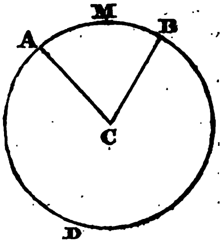
For if $\frac{1}{2}$CA×*circ.* CA is not the area of the circle whose radius is CA, it must be the area of a circle either greater or less. Let us first suppose it to be the area of a greater circle; and, if possible, that $\frac{1}{2}$CA×*circ.* CA$=$ *surf.* CB.

About the circle whose radius is CA describe a regular polygon DEFG &c., such (10. IV.) that its sides shall not meet the circumference whose radius is CB. The surface of thispolygon will be equal (7. IV.) to its perimeter DE+EF+FG+ &c. multiplied by $\frac{1}{2}$AC: but the perimeter of the polygon is greater than the inscribed circumference enveloped by it on all sides; hence the surface of the polygon DEFG &c. is greater than $\frac{1}{2}$AC× *circ.* AC, which by the supposition is the measure of the circle whose radius is CB; thus the polygon must be greater than that circle. But in reality it is less, being contained wholly within the circumference: hence it is impossible that $\frac{1}{2}$CA× *circ.* AC can be greater than *surf.* CA; in other words, it is impossible that the circumference of a circle multiplied by half its radius can be the measure of a greater circle.

In the second place, we assert it to be equally impossible that this product can be the measure of a smaller circle. To avoid the trouble of changing our figure, let us suppose that the circle in question is the one whose radius is CB: we are to shew that $\frac{1}{2}$CB×*circ.* CB cannot be the measure of a smaller circle, of the circle, for instance, whose radius is CA. Grant it to be so; and that, if possible, $\frac{1}{2}$CB×*circ.* CB$=$*surf.* CA.

Having made the same construction as before, the surface of the polygon DEFG, &c. will be measured by (DE+EF+FG+ &c.) ×$\frac{1}{2}$CA; but the perimeter DE+EF+FG+ &c. is less than *circ.* CB, being enveloped by it on all sides; hence the area of the polygon is less than $\frac{1}{2}$CA× *circ.* CB, and still more than $\frac{1}{2}$CB × *circ.* CB. Now by the supposition, this last quantity is the measure of the circle whose radius is CA: hence the polygon must be less than the inscribed circle, which is absurd; hence it is impossible that the circumference of a circle multiplied by half its radius, can be the measure of a smaller circle.

Hence, finally, the circumference of a circle multiplied by half its radius is the measure of that circle itself.



Cor. 1. The surface of a sector is equal to the arc of that sector multiplied by half its radius.

For, the sector ACB (17. II.) is to the whole circle as the arc AMB is to the whole circumference ABD, or as $\text{AMB}\times\frac{1}{2}\text{AC}$ is to $\text{ABD}\times\frac{1}{2}\text{AC}$. But the whole circle is equal to $\text{ABD}\times\frac{1}{2}\text{AC}$; hence the sector ACB is measured by $\text{AMB}\times\frac{1}{2}\text{AC}$.

Cor. 2. Let the circumference of the circle whose diameter is unity be denoted by π: then, because circumferences are to each other as their radii or diameters, we shall have the diameter 1 to its circumference π as the diameter 2CA is to the circumference whose radius is CA, that is, $1 : \pi :: 2\text{CA} : \textit{circ.}\ \text{CA}$, therefore $\textit{circ.}\ \text{CA}=2\pi\times\text{CA}$. Multiply both terms by $\frac{1}{2}\text{CA}$; we have $\frac{1}{2}\text{CA}\times\textit{circ.}\ \text{CA}=\pi\times\text{CA}^2$, or $\textit{surf.}\ \text{CA}=\pi\times\text{CA}^2$: hence the surface of a circle is equal to the product of the square of its radius by the constant number π, which represents the circumference whose diameter is 1, or the ratio of the circumference to the diameter.

In like manner, the surface of the circle, whose radius is OB, will be equal to $\pi\times\text{OB}^2$; but $\pi\times\text{CA}^2 : \pi\times\text{OB}^2 :: \text{CA}^2 : \text{OB}^2$; hence *the surfaces of circles are to each other as the squares of their radii*, which agrees with the preceding theorem.

Scholium. We have already observed, that the problem of the quadrature of the circle consists in finding a square equal in surface to a circle, the radius of which is known. Now it has just been proved, that a circle is equivalent to the rectangle contained by its circumference and half its radius; and this rectangle may be changed into a square, by finding (6. III.) a mean proportional between its length and its breadth. To square the circle, therefore, is to find the circumference when the radius is given; and for effecting this, it is enough to know the ratio of the circumference to its radius or its diameter.

Hitherto, the ratio in question has never been determined except approximately; but the approximation has been carried so far, that a knowledge of the exact ratio would afford no real advantage whatever beyond that of the approximate ratio. Accordingly, this problem, which engaged geometers so deeply, when their methods of approximation were less perfect, is now degraded to the rank of those idle questions, with which no one arrived at this point, we shall infer that the last result expresses the surface of the circle, which, since it must always lie between the inscribed and the circumscribed polygon, and since those polygons agree as far as a certain place of decimals, must also agree with both as far as the same place.

We have subjoined the computation of those polygons, carried on till they agree as far as the seventh place of decimals.

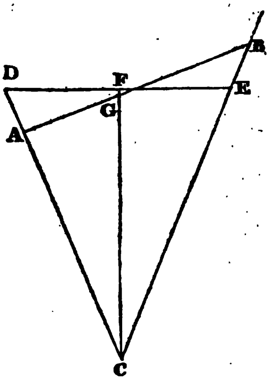
Number of sides.	Inscribed polygon.	Circumscribed polygon.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441184
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928
16384	3.1415925	3.1415927
32768	3.1415926	3.1415926

The surface of the circle, we infer therefore, is equal to 3.1415926. Some doubt may exist perhaps about the last decimal figure, owing to errors proceeding from the parts omitted; but the calculation has been carried on with an additional figure, that the final result here given might be absolutely correct even to the last decimal place.

Since the surface of the circle is equal to half the circumference multiplied by the radius, the half circumference must be 3.1415926, when the radius is 1; or the whole circumference must be 3.1415926, when the diameter is 1: hence the ratio of the circumference to the diameter, formerly expressed by π, is equal to 3.1415926.

PROPOSITION XV. LEMMA.

The triangle CAB *is equivalent to the isosceles triangle* DCE, *which has the same angle* C, *and one of its equal sides* CE *or* CD *a mean proportional between* CA *and* CB. *And if the angle* CAB *is right, the perpendicular* CF, *drawn to the base of the isosceles triangle, will be a mean proportional between the side* CA *and half the sum of the sides* CA, CB.



First. Because of the common angle C, the triangle ABC is to the isosceles triangle DCE as $AC\times CB$ is to $DC\times CE$ or DC^2 (24. III.): hence those triangles will be equivalent, if $DC^2=AC\times CB$, or if DC is a mean proportional between AC and CB.

Secondly. Because the perpendicular CGF bisects the angle ACB, we shall have $AG : GB :: AC : CB$ (17. III.); and therefore, *componendo*, $AG : AG+GB$ or $AB :: AC : AC+CB$; but AG is to AB as the triangle ACG is to the triangle ACB, or 2 CDF; besides if the angle A is right, the right-angled triangles ACG, CDF must be similar, and give $ACG : CDF :: AC^2 : CF^2$; hence,

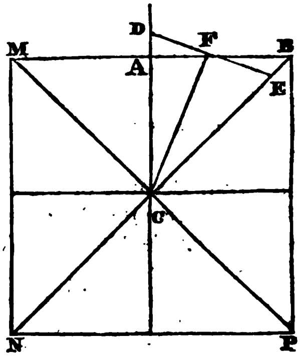
$$AC^2 : 2CF^2 :: AC : AC+CB.$$

Multiply the second pair by AC; the antecedents will be equal, and consequently we shall have $2CF^2=AC.(AC+CB)$, or $CF^2=AC.\left(\frac{AC+CB}{2}\right)$; hence if the angle A is right, the perpendicular CF will be a mean proportional between the side AC and the half sum of the sides AC, CB.

PROPOSITION XVI. PROBLEM.

To find a circle differing as little as we please from a given regular polygon.

Let the square BMNP be the proposed polygon. From the centre C, draw CA perpendicular to MB, and join CB.



The circle described with the radius CA is inscribed in the square, and the circle described with the radius CB circumscribes this same square; the first will in consequence be less than it, the second greater: it is now required to compress those limits.

Take CA and CE, each equal to the mean proportional between CA and CB, and join ED; the isosceles triangle CDE will, by the last proposition, be equivalent to the triangle CAB. Perform the same operation on each of the eight triangles which compose the square: you will thus form a regular octagon equivalent to the square BMNP. The circle described with the radius CF, a mean proportional between CA and $\frac{\mathrm{CA}+\mathrm{CB}}{2}$, will be inscribed in this octagon, and the circle whose radius is CD will circumscribe it. The first of them will therefore be less than the given square, the second greater.

If the right-angled triangle CDF be, in like manner, changed into an equivalent isosceles triangle, we shall by this means form a regular polygon of 16 sides, equivalent to the proposed square. The circle inscribed in this polygon will be less than the square; the circumscribed circle will be greater.

The same process may be continued, till the ratio between the radius of the inscribed and that of the circumscribed circle, approach as near to equality as we please. In that case, both circles may be regarded as equivalent to the square.

Scholium. The investigation of the successive radii is reduced to this. Let a be the radius of the circle inscribed in one of the polygons, b the radius of the circle circumscribing the same polygon; let a' and b' be the corresponding radii for the next polygon, which is to have twice the number of sides. From what has been demonstrated, b' is a mean proportional between a and b, and a' is a mean proportional between a and $\frac{a+b}{2}$; so that $b'=\surd a.b$, and $a'=\surd a.\frac{a+b}{2}$: hence a and b the radii of one polygon being known, we may easily discover the radii a' and b' of the next polygon; and the process may be continued till the difference between the two radii become insensible; then either of those radii will be the radius of the circle equivalent to the proposed square or polygon.

This method is easily practised with regard to lines; for it implies nothing but the finding of successive mean proportionals between lines which are given: it is still more easily practised with regard to numbers, and forms one of the most commodious plans which elementary geometry can furnish, for discovering speedily the approximate ratio of the circumference to the diameter. Let the side of the square be 2; the first inscribed radius CA will be 1, and the first circumscribed radius CB will be $\surd 2$ or 1.4142136. Hence, putting $a=1$, $b=1.4142136$, we shall find $b'=1.1892071$, and $a'=1.0986841$. These numbers will serve for computing the rest, the law of their combination being known.

Radii of the circumscribed circles.	Radii of the inscribed circles.
1.4142136	1.0000000
1.1892071	1.0986841
1.1430500	1.1210863
1.1320149	1.1265639
1.1292862	1.1279257
1.1286063	1.1282657

Since the first half of these ciphers is now become the same on both sides, it will occasion little error to assume the arithmetical means instead of the mean proportionals or geometrical means, which differ from the former only in their last figures. By this method, the operation is greatly abridged; the results are:

1.1284360	1.1283508
1.1283934	1.1283721
1.1283827	1.1283774
1.1283801	1.1283787
1.1283794	1.1283791
1.1283792	1.1283792

Thus 1.1283792 is very nearly the radius of a circle equal in surface to the square whose side is 2. From this, it is easy to find the ratio of the circumference to the diameter: for it has already been shewn that the surface of the circle is equal to the square of its radius multiplied by the number π; hence if the

surface 4 be divided by the square of 1.1283792 the radius, we shall get the value of π, which by this computation is found to be 3.1415926, &c., as was formerly determined by another method.

APPENDIX TO BOOK IV.

DEFINITIONS.

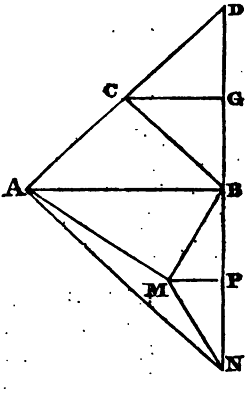
I. A *maximum* is the greatest among all the quantities of the same species; a *minimum* is the least.

Thus the diameter of a circle is a *maximum* among all the lines joining two points in the circumference; the perpendicular is a *minimum* among all the lines drawn from a given point to a given straight line.

II. *Isoperimetrical figures* are such as have equal perimeters.

PROPOSITION I. THEOREM.

Of all the triangles having the same base and the same perimeter, the maximum *is that triangle of which the two undetermined sides are equal.*



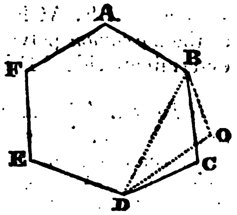
Suppose AC=CB, and AM+MB=AC+CB; then is the isosceles triangle ACB greater than the triangle AMB, which has the same base and the same perimeter.

From the centre C, with a radius CA=CB, describe a circle meeting CA produced in D; join DB: the angle DBA, inscribed in a semicircle, will be right (15. II.). Produce the perpendicular DB towards N, make MN=MB, and join AN. Lastly, from the points M and C, draw MP and CG perpendicular to DN.

Since CB=CD, and MN=MB, we have AC+CB=AD, and AM+MB=AM+MN. But AC+CB=AM+MB; therefore AD=AM+MN; therefore AD$>$AN: and since the oblique line AD is greater than the oblique line AN, it must be farther from the perpendicular AB; therefore DB$>$BN; therefore BG, which (12. I.) is half of BD, will be greater than BP which is half of BN. But the triangles ABC, ABM, having the same base AB, are to each other as their altitudes BG, BP; therefore, since BG$>$BP, the isosceles triangle ABC is greater than ABM, which is not isosceles, and has the same base and the same perimeter.

PROPOSITION II. THEOREM.

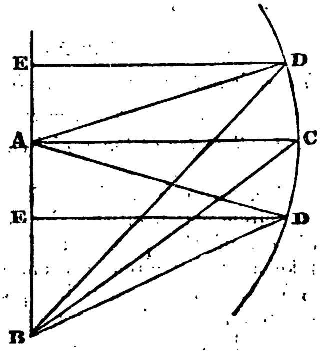
Of all the isoperimetrical polygons having a given number of sides, the maximum *is the one which has its sides equal.*



For, let ABCDEF be the *maximum* polygon. If the side BC is not equal to CD, construct upon the base BD an isosceles triangle BOD, which shall be isoperimetrical with BCD; this triangle BOD will (Prop. I. Ap.) be greater than BCD, and consequently the polygon ABODEF will be greater than ABCDEF; hence the latter is not the *maximum* of all the polygons having the same perimeter and the same number of sides, which contradicts the hypothesis. BC must therefore be equal to CD: for the same reasons must CD=DE, DE=EF, &c.; hence all the sides of the *maximum* polygon are equal.

PROPOSITION III. THEOREM.

Of all the triangles, having two sides given in length and containing an angle which is not given, the maximum *is that triangle in which the two given sides contain a right angle.*

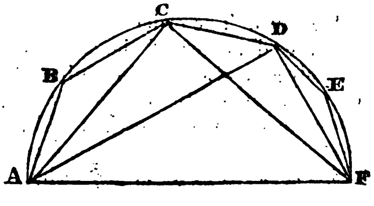


Let BAC, BAD be two triangles, in which the side AB is common, and the side AC=AD; if the angle BAC is right, the triangle BAC will be greater than the triangle BAD, of which the angle A is acute or obtuse.

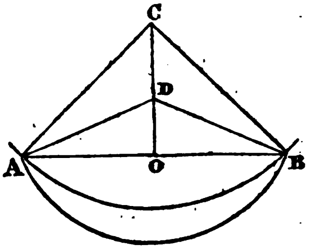
For, the base AB being the same, the two triangles BAC, BAD are to each other as their altitudes AC, DE; but the perpendicular DE is shorter than the oblique line AD or its equal AC; hence the triangle BAD is less than the triangle BAC.

PROPOSITION IV. THEOREM.

Among polygons formed of sides which are all given but one, the maximum *is such that all its angles can be inscribed in a semicircle, of which the unknown side is the diameter.*



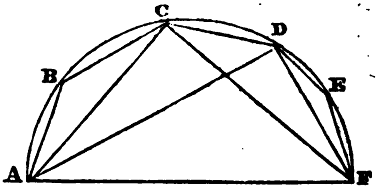
Let ABCDEF be the greatest polygon which can be formed with the given sides AB, BC, CD, DE, EF, and the last side AF assumed at pleasure. Draw the diagonals AD, DF. If the angle ADF were not right, then by making it right we should augment the triangle ADF (Prop. 3. Ap.); and consequently augment the whole polygon, because the parts ABCD, DEF would continue exactly as they are. But this polygon, being already a *maximum*, cannot be augmented; hence the angle ADF is no other than a right angle. The same are ABF, ACF, AEF; hence all the angles A, B, C, D, E, F of the *maximum* polygon are in a semicircumference, of which the indeterminate side AF is the diameter.



Scholium. This proposition gives rise to a question: Whether there be more ways than one of forming a polygon with sides which are all given, except the last side which is unknown and is to form the diameter of the semicircle wherein all the others are inscribed? Before deciding this question, it will be necessary to observe, that if the same chord AB subtend two arcs described with different radii AC, AD, the central angle standing upon this chord, will be smaller in the circle whose radius is greater; thus $ACB \angle ADB$. For (27. I.), the angle $ADO = ACD + CAD$; hence $ACD \angle ADO$, and doubling both, $ACB \angle ADB$.

PROPOSITION V. THEOREM.

There is but one way of forming a polygon ABCDEF *with sides which are all given, except the last side, which is unknown, and is to form the diameter of the semicircle wherein all the others are inscribed.*

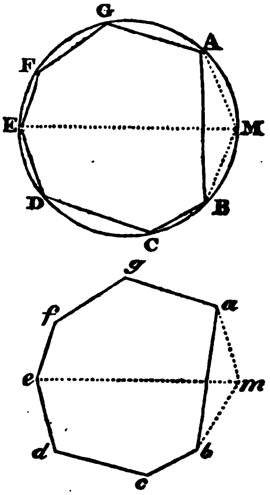


For, suppose we have found one circle which satisfies the conditions of the problem: if we take a greater circle, the chords AB, BC, CD, &c. will lie opposite to angles at the centre, which are smaller. Hence the sum of these central angles will be less than two right angles; hence the extremities of the given sides will not fall at the extremities of a diameter. The contrary error will arise, if we assume a smaller circle: hence the polygon in question can only be inscribed in one circle.

Scholium. The order of the sides AB, BC, CD may be altered at will, the diameter of the circumscribed circle, as well as the area of the polygon, always continuing the same; for whatever be the order of the arcs AB, BC, &c., it is enough if their sum be a semicircumference, and the polygon will always have the same area, being always equal to the semicircle *minus* the segments AB, BC &c. the sum of which is the same in any order.

PROPOSITION VI. THEOREM.

Of all the polygons formed with given sides, the maximum *is the one which can be inscribed in a circle.*



Let ABCDEFG be the inscribed polygon, and *abcdefg* the polygon which cannot be inscribed, both having equal sides, AB=*ab*, BC=*bc*, &c.; the inscribed polygon will be greater than the other.

Draw the diameter EM; join AM, MB; upon *ab*=AB, construct the triangle *abm* equal to ABM, and join *em*.

By Prop. 4. of this Appendix, the polygon EFGAM, is greater than *efgam*, unless this *efgam* can be inscribed in a semicircle, of which the side *em* is the diameter, in which case the two polygons would be equal, by the last Proposition. For the same reason, the polygon EDCBM is greater than *edcbm*, saving a similar exception, by means of which they would be equal. Hence the whole polygon EFGAMBCDE is greater than *efgambcde*, unless they are equal in all respects: but they are not equal in all respects, because the one is inscribed in a circle, and the other cannot be inscribed; hence the inscribed polygon is greater. Take away from both respectively the equal triangles ABM, *abm*; there will remain the inscribed polygon ABCDEFG greater than *abcdefg*, which cannot be inscribed.

Scholium. It might be shewn, as in the foregoing Proposition, that there can be only one circle, and therefore only one *maximum* polygon that will satisfy the conditions of the problem; and this polygon will always have the same area, in whatever order we arrange its sides.

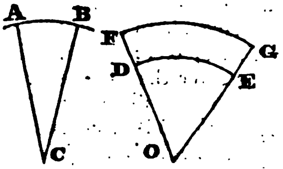
PROPOSITION VII. THEOREM.

The regular polygon is the greatest of all the polygons which have the same perimeter and the same number of sides.

For, by the second of these Theorems, the *maximum* polygon has all its sides equal; and by the last Theorem, it can be inscribed in a circle: hence it is a regular polygon.

PROPOSITION VIII. LEMMA.

Two angles at the centre, measured in two different circles, are to each other as their included arcs divided by their radii.



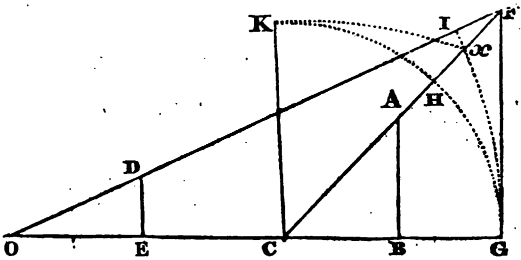
The angle C is to the angle O as the quotient $\frac{AB}{AC}$ is to the quotient $\frac{DE}{DO}$.

With a radius OF, equal to AC, describe the arc FG included between the sides OD, OE produced. By reason of the equal radii AC, OF, (17. II.) we shall have $C : O :: AB : FG$; hence $C : O :: \frac{AB}{AC} : \frac{FG}{FO}$. But by reason of the similar arcs FG, DE, (11. IV.) we have $FG : DE :: FO : DO$; therefore the quotient $\frac{FG}{FO}$ is equal to the quotient $\frac{DE}{DO}$, and consequently $C : O :: \frac{AB}{AC} : \frac{DE}{DO}$.

PROPOSITION IX. THEOREM.

Of two isoperimetrical regular polygons, the one having the greater number of sides is the greater.

Let DE be a half-side of one of those polygons, O the centre, OE the apothem: let AB be a half-side of the other polygon, C the centre, CB the apothem. Suppose the centres O and C to be situated at any distance OC, and the apothems OE, CB, in the direction OC: thus



DOE and ACB will be half-angles at the centres of the polygons; and because these angles are not equal, the lines CA, OD, if produced, will meet in some point F; from this point let fall the perpendicular OC as FG; from the points O and C as centres, describe the arcs GI, GH, terminated by the sides OF, CF.

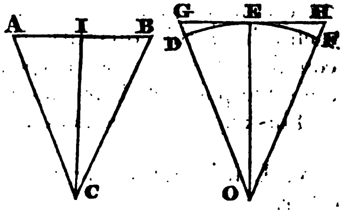
Now, by the preceding lemma, we have O : C :: $\frac{GI}{OG}$: $\frac{GH}{CG}$; but DE is to the perimeter of the first polygon as the angle O is to four right angles; and AB is to the perimeter of the second polygon as C is to four right angles; therefore, since the perimeters of the polygons are equal, DE : AB :: O : C, or DE : AB :: $\frac{GI}{OG}$: $\frac{GH}{CG}$. Multiply the antecedents by OG, and the consequents by CG; we shall have DE.OG : AB.CG :: GI : GH. But the similar triangles ODE, OFG give OE : OG :: DE : FG, whence DE.OG=OE.FG; in like manner, we should find AB.CG=CB.FG; therefore OE.FG : CB.FG :: GI : GH, or OE : CB :: GI : GH. Hence, if we can shew that the arc GI is greater than the arc GH, it will follow that the apothem OE is greater than CB.

On the other side of CF, construct a figure CK*x* entirely equal to the figure CG*x*, so that CK=CG, the angle HCK=HCG, and the arc K*x*=*x*G; the curve K*x*G will envelope the arc KHG, and (9. IV.) be greater than it. Therefore G*x*, half of the curve, is greater than GH half of the arc; therefore still more is GI greater than GH.

It follows, therefore, that the apothem OE is greater than CB; but (7. IV.) the two polygons, having the same perimeter, are to each other as their apothems; hence the polygon which has DE for a half-side is greater than the polygon which has AB for its half-side: the first has more sides, because its central angle is smaller; hence of two isoperimetrical regular polygons, the one having more sides is greater.

PROPOSITION X. THEOREM.

The circle is greater than any polygon of the same perimeter.



We have already shewn, that of all the isoperimetrical polygons having the same number of sides, the regular polygon is the greatest; therefore we need only to compare the circle with some regular polygon of the same perimeter. Let AI be the half-side of this polygon; C its centre. In the isoperimetrical circle, let the angle DOE be equal to ACI, and onsequently, the arc DE be equal to the half-side AI. The polygon P is to the circle C as the triangle ACI is to the sector ODE; hence P : C :: ½AI.CI : ½DE.OE :: CI : OE. From the point E draw a tangent EG meeting OD produced in G: the similar triangles ACI, GOE will give the proportion CI : OE :: AI or DE : GE; hence P : C :: DE : GE, or as DE.½OE,

which is the measure of the sector DOE, is to GE.$\frac{1}{2}$OE, which is the measure of the triangle GOE : now this sector is less than the triangle; hence P is less than C; hence the circle is greater than any isoperimetrical polygon.

BOOK V.

PLANES AND SOLID ANGLES.

Definitions.

I. A straight line is *perpendicular to a plane*, when it is perpendicular to all the straight lines (4. IV.) which pass through its *foot* in the plane. Conversely, the plane is perpendicular to the line.

The *foot* of the perpendicular is the point at which that line meets the plane.

II. A line is *parallel to a plane*, when it cannot meet that plane, to whatever distance both be produced. Conversely, the plane is parallel to the line.

III. Two *planes* are *parallel* to each other, when they cannot meet, to whatever distance both be produced.

IV. It will be demonstrated (3. IV.) that the common intersection of two planes which meet each other is a straight line: that granted, the *angle* or mutual *inclination of two planes* is the quantity, greater or less, by which they are separated from each other; this quantity is measured (7. IV.) by the angle contained between two perpendiculars drawn in these planes, at the same point of their common intersection.

This angle may be acute, or right, or obtuse.

V. If it is right, the two *planes* are perpendicular to each other.

VI. A *solid angle* is the angular space included between several planes which meet at the same point.

Thus, the solid angle (see the fig. of Prop. 25. V. Schol.) is formed by the union of the planes ASB, BSC, CSD, DSA.

Three planes at least are requisite to form a solid angle.

PROPOSITION I. THEOREM.

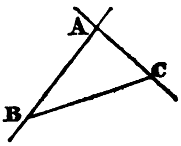
A straight line cannot be partly in a plane, and partly out of it.

For, by the definition of a plane, when a straight line has two points common with a plane, it lies wholly in that plane.

Scholium. To discover whether a surface is plane, it is necessary to apply a straight line in different ways to that surface, and to observe if it touches the surface throughout its whole extent.

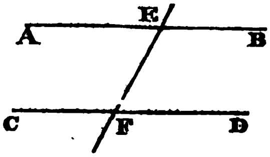
PROPOSITION II. THEOREM.

Two straight lines, which intersect each other, lie in the same plane, and determine its position.



Let AB, AC be two straight lines which intersect each other in A; a plane may be conceived in which the straight AB is found; if this plane be turned round AB, until it pass through the point C, then the line AC, which has two of its points A and C in this plane, lies wholly in it; hence the position of the plane is determined by the single condition of containing the two straight lines AB, AC.

Cor 1. A triangle ABC, or three points A, B, C not in a straight line, determine the position of a plane.



Cor. 2. Hence also two parallels AB, CD determine the position of a plane; for, drawing the secant EF, the plane of the two straight lines AE, EF is that of the parallels AB, CD.

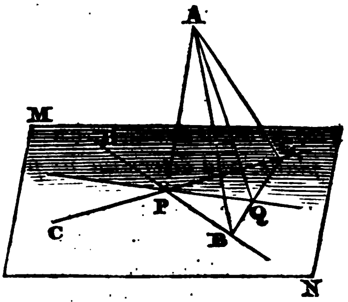
PROPOSITION III. THEOREM.

If two planes cut each other, their common intersection will be a straight line.

For, if among the points common to the two planes, there be three which are not in the same straight line, then the planes, passing each through these three points, must form only one and the same plane; which contradicts the hypothesis.

PROPOSITION IV. THEOREM.

If a straight line AP *is perpendicular to two other straight lines* PB, PC, *which cross each other at its foot in the plane* MN, *it will be perpendicular to any straight line* PQ *drawn through its foot in the same plane, and thus it will be perpendicular to the plane* MN.



Through any point Q in PQ, draw (Prob. 5. III.) the straight line BC in the angle BPC, so that BQ=QC; join AB, AQ, AC.

The base BC being divided into two equal parts at the point L, the triangle BPC (14. III.) will give

$$PC^2+PB^2=2PQ+2QC^2.$$

The triangle BAC will in like manner give

$$AC^2+AB^2=2AQ^2+2QC^2.$$

Taking the first equation from the second, and observing that the triangles APC, APB, which are both right-angled at P, give $AC^2-PC^2=AP^2$, and $AB^2-PB^2=AP^2$; we shall have

$$AP^2+AP^2=2AQ^2-2PQ^2.$$

Therefore, by taking the halves of both, we have $AP^2=AQ^2-PQ^2$, or $AQ^2=AP^2+PQ^2$; hence the triangle APQ is right-angled at P; hence AP is perpendicular to PQ.

Scholium. Thus it is evident, not only that a straight line may be perpendicular to all the straight lines which pass through its foot in a plane, but that it always must be so, whenever it is perpendicular to two straight lines drawn in the plane; which proves our first Definition to be accurate.

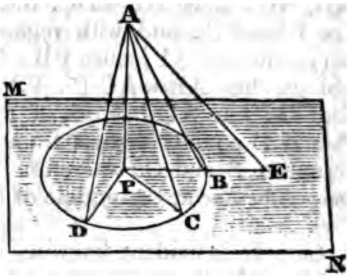
Cor. 1. The perpendicular AP is shorter than any oblique line AQ; therefore it measures the true distance from the point A to the plane PQ.

Cor. 2. At a given point P on a plane, it is impossible to erect more than one perpendicular to that plane; for if there could be two perpendiculars at the same point P, draw along these two perpendiculars a plane, whose intersection with the plane MN is PQ; then those two perpendiculars would be perpendicular to the line PQ, at the same point, and in the same plane, which is impossible.

It is also impossible to let fall from a given point out of a plane two perpendiculars to that plane; for let AP, AQ, be these two perpendiculars, then the triangle APQ would have two right angles APQ, AQP, which is impossible.

PROPOSITION V. THEOREM.

Oblique lines equally distant from the perpendicular are equal; and, of two oblique lines unequally distant from the perpendicular, the more distant is the longer.



For the angles APB, APC, APB, being right, if we suppose the distances PB, PC, PD, to be equal to each other, the triangles APB, APC, APD, will have each an equal angle contained by equal sides; therefore they will be equal; therefore the hypotenuses or the oblique lines AB, AC, AD, will be equal to each other. In like manner, if the distance FE is greater than PD or its equal PB, the oblique line AE will evidently be greater than AB, or its equal AD.

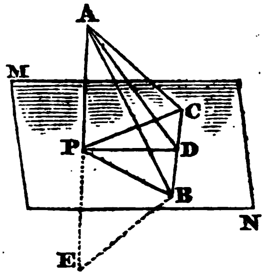
Cor. All the equal oblique lines AB, AC, AD, &c. terminate in the circumference BCD, described from P the foot of the perpendicular as a centre; therefore a point A being given out of a plane, the point P at which the perpendicular let fall from A would meet that plane, may be found by marking upon that plane three points B, C, D, equally distant from the point A, and then finding the centre of the circle which passes through these points; this centre will be P, the point sought.

Scholium. The angle ABP is called the *inclination of the oblique line* AB *to the plane* MN; which inclination is evidently equal with respect to all such lines AB, AC, AD, as are equally distant from the perpendicular; for all the triangles ABP, ACP, ADP, &c. are equal to each other.

PROPOSITION VI. THEOREM.

Let AP *be a perpendicular to the plane* MN, *and* BC *a line situated in that plane; if from* P *the foot of the perpendicular,* PD *is drawn at right angles to* BC, *and* AD *is joined,* AD *will be perpendicular to* BC.

Take DB=DC, and join PB, PC, AB, AC; since DB=DC, the oblique line PB=PC: and with regard to the perpendicular AP, since PB=PC, the oblique line AB=AC (5. V.); therefore the line AD has two of its points A and D equally distant from the extremities B and C; therefore AD is a perpendicular at the middle of BC.



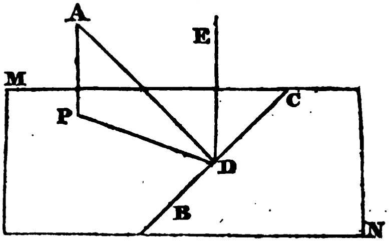
Cor. It is evident likewise, that BC is perpendicular to the plane APD, since BC is at once perpendicular to the two straight lines AD, PD.

Scholium. The two straight lines AE, BC afford an instance of two lines which do not meet, because they are not situated in the same plane. The shortest distance between these lines is the straight line PD, which is at once perpendicular to the line AP and to the line BC. The distance PD is the shortest between these two lines; for if we join any other two points, such as A and B, we shall have AB>AD, AD>PD; therefore AB>PD.

The two lines AE, CB, though not situated in the same plane, are conceived as forming a right angle with each other, because AD and the line drawn through one of its points parallel to BC would make with each other a right angle. In the same manner, the line AB and the line PD, which represent any two straight lines not situated in the same plane, are supposed to form with each other the same angle, which would be formed by AB and a straight line parallel to PD drawn through one of the points of AB.

PROPOSITION VII. THEOREM.

If the line AP *is perpendicular to the plane* MN, *any line* DE *parallel to* AP *will be perpendicular to the same plane.*



Along the parallels AP, DE, extend a plane; its intersection with the plane MN will be PD; in the plane MN draw BC perpendicular to PD, and join AD.

By the Corollary of the preceding Theorem, BC is perpendicular to the plane APDE; therefore the angle BDE is right: but the angle EDP is right also, since AP is perpendicular to PD, and DE parallel to AP; therefore the line DE is perpendicular to the two straight lines DP, DB; therefore it is perpendicular to their plane MN.

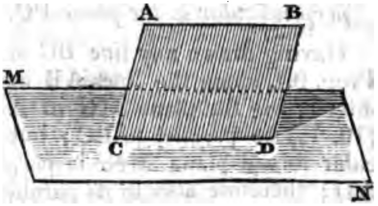
Cor. 1. Conversely, if the straight lines AP, DE are perpendicular to the same plane MN, they will be parallel; for if they be not so, draw through the point D a line parallel to AP, this parallel will be perpendicular to the plane MN; therefore through the same point D more than one perpendicular might be erected in the same plane, which (4. V.) is impossible.

Cor. 2. Two lines A and B, parallel to a third C, are parallel to each other; for, conceive a plane perpendicular to the line C; the lines A and B, being parallel to C, will be perpendicular to the same plane; therefore, by the preceding Corollary, they will be parallel to each other.

The three lines are understood not to be in the same plane; otherwise the proposition (24. I.) would be already known.

PROPOSITION VIII. THEOREM.

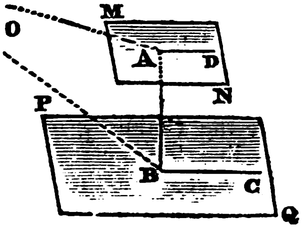
If the line AB *is parallel to a straight line* CD *drawn in the plane* MN, *it will be parallel to that plane.*



For if the line AB, which lies in the plane ABCD, could meet the plane MN, this could only be in some point of the line CD, the common intersection of the two planes: but AB cannot meet CD, since they are parallel; hence it will not meet the plane MN; hence (*Def.* 2.) it is parallel to that plane.

PROPOSITION IX. THEOREM.

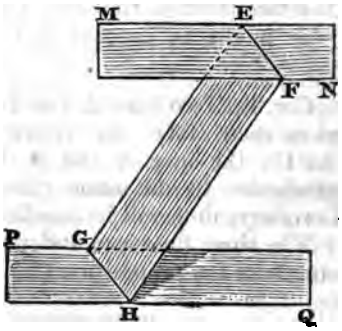
Two planes MN, PQ *perpendicular to the same straight line* AB, *are parallel to each other.*



For, if they can meet anywhere, let O be one of their common points, and join OA, OB; the line AB which is perpendicular to the plane MN, is perpendicular to the straight line OA drawn through its foot in that plane; for the same reason AB is perpendicular to BO; therefore OA and OB are two perpendiculars let fall, from the same point O, upon the same straight line; which is impossible: therefore the planes MN, PQ, cannot meet each other; therefore they are parallel.

PROPOSITION X. THEOREM.

The intersections EF, GH, *of two parallel planes* MN, PQ *with a third plane* FG, *are parallel.*



For, if the lines EF, GH, lying in the same plane, were not parallel, they would meet each other when produced; therefore the planes MN, PQ, in which those lines lie, would also meet; therefore the planes would not be parallel.

PROPOSITION XI. THEOREM.

The line AB, *which is perpendicular to the plane* MN, *is also perpendicular to the plane* PQ, *parallel to* MN.

Having drawn any line BC in the plane PQ (see the fig. of Prop. 9.), along the lines AB and BC, extend a plane ABC, intersecting the plane MN in AD; the intersection AD will (10. V.) be parallel to BC; but the line AB, being perpendicular to the plane MN, is perpendicular to the straight line AD; therefore also to its parallel BC: hence the line AB being perpendicular to any line BC drawn through its foot in the plane PQ, is consequently perpendicular to that plane.

PROPOSITION XII. THEOREM.

The parallels EG, FH, *comprehended between two parallel planes* MN, PQ, *are equal.*

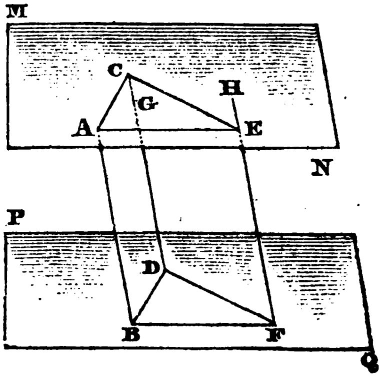
Through the parallels EG, FH, (see the fig. of Prop. 10.) draw the plane EGHF to meet the parallel planes in EF and GH. The intersections EF, GH, (10. V.) are parallel to each other; so likewise are EG, FH; therefore the figure EGHF is a parallelogram; therefore EG=FH.

Cor. Hence it follows that *two parallel planes are every where equidistant;* for if EG and FH are perpendicular to the two planes MN, PQ, they will be parallel to each other, (7. V. Cor. 1.); and therefore equal, as has just been shewn.

PROPOSITION XIII. THEOREM.

If two angles CAE, DBF *not situated in the same plane, have their sides parallel and lying in the same direction, those angles will be equal and their planes will be parallel.*

Make AC=BD, AE=BF; and join CE, DF, AB, CD, EF. Since AC is equal and parallel to BD, the figure ABDC is a parallelogram (Prop. 31. I.); therefore CD is equal and parallel to AB. For a similar reason, EF is equal and parallel to AB; hence also CD is equal and parallel to EF; hence the figure CEFD is a parallelogram, and the side CE is equal and parallel to DF; therefore the triangles CAE, DBF have their corresponding sides equal; therefore the angle CAE=DBF.



Again, the plane ACE is parallel to the plane BDF. For suppose the plane parallel to BDF, drawn through the point A, were to meet the lines CD, EF, in points different from C and E, for instance in G and H; then, (12. V.), the three lines AB, GD, FH would be equal: but the lines AB, CD, EF are already known to be equal; hence CD=GD, and FH=EF, which is absurd; hence the plane ACE is parallel to BDF.

Cor. If two parallel planes MN, PQ are met by two other planes CADB, EABF, the angles CAE, DBF, formed by the intersections of the parallel planes will be equal; for (10. V.) the intersection AC is parallel to BD, and AE to BF, therefore the angle CAE=DBF.

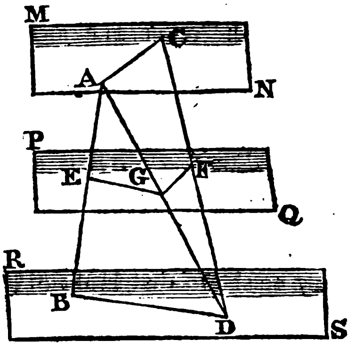
PROPOSITION XIV. THEOREM.

If three straight lines AB, CD, EF, *not situated in the same plane, are equal and parallel, the triangles* ACE, BDF *formed by joining the extremities of these straight lines will be equal, and their planes will be parallel.*

For, since AB (see the preceding figure), is equal and parallel to CD, the figure ABDC is a parallelogram; hence the side AC is equal and parallel to BD. For a like reason the sides AE, BF are equal and parallel, as also CE, DF; therefore the two triangles ACE, BDF, are equal; and consequently, as in the last Proposition, their planes are parallel.

PROPOSITION XV. THEOREM.

Two straight lines, included between three parallel planes, are cut proportionally.



Suppose the line AB to meet the parallel planes MN, PQ, RS, at the points A, E, B; and the line CD to meet the same planes at the points C, F, D: we are now to shew that AE : EB : : CF : FD.

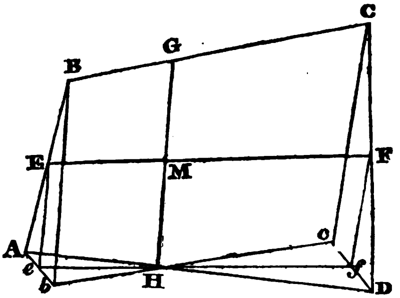
Draw AD meeting the plane PQ in G, and join AC, EG, GF, BD; the intersections EG, BD, of the parallel planes PQ, RS, in the plane ABD, are parallel (10. V.); therefore AE : EB : : AG : GD; in like manner, the intersections AC, GF being parallel, AG : GD : : CF : FD; the ratio AG : GD is the same in both; hence

AE : EB : : CF : FD.

PROPOSITION XVI. THEOREM.[*]

Let ABCD *be any quadrilateral situated or not situated wholly in one plane ; if two straight lines* EF, GH *cut the opposite sides proportionally, so that* AE : EB : : DF : FC, *and* BG : GC :: AH : HD ; *then will the straight lines* EF, GH, *cut each other in such a manner that we shall have* HM : MG :: AE : EB, *and* EM : MF : : AH : HD.

Draw along AD any plane A*b*H*c*D, which shall not lie along GH; through the points E, B, C, F, draw E*e*, B*b*, C*c*, F*f*, parallel to GH, to meet that plane in *e*, *b*, *c*, *f*. Because B*b*, GH, C*c*, are parallel, (15. III.) we have *b*H : H*c* : : BG : GC :: AH: HD; therefore (20. III.) the triangles AH*b*, DH*c* are similar. Also we have A*e* : *eb* : : AE : EB, and D*f* : *fc* :: DF : FC; therefore A*e* : *eb* : : D*f* : *fc*, or *componendo*, A*e* : D*f* : : A*b* : D*c* ; but, because the triangles AH*b*, DH*c*, are similar, we have A*b* : D*c* : : AH : HD; therefore A*e* : D*f* : : AH : HD; also the triangles AH*b*, *c*HD being similar, the angle HA*e*=HD*f*; therefore (20. III.) the triangles AH*e*, DH*f* are similar; therefore the angle AH*e*=DH*f*. Hence it follows that *e*H*f* is a straight line, and therefore the three parallels E*e*, GH, F*f* are situated in the same plane, which plane will contain the two straight lines EF, GH; therefore *these latter must cut each other in a point* M. Lastly, because E*e*, MH, F*f* are parallel, we have EM : MF : : *e*H : H*f* :: AH : HD.



And, by a similar construction in reference to the side AB, it may be shewn that HM : MG : : AE : EB.

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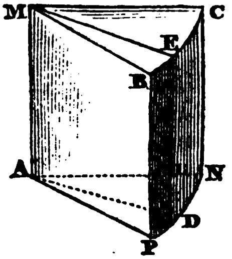
[*] May be omitted, having no immediate connexion with what follows.
ED.

PROPOSITION XVII. THEOREM.

The angle included between the two planes MAN, MAP *may be measured, agreeably to our Definition, by the angle* NAP, *which is formed by the two perpendiculars* AN, AP *drawn in each of those planes to the common intersection* AM.



To shew the legitimacy of this measurement, we must, in the first place prove that it is constant, or that it would be the same at whatever point of the common intersection the perpendiculars were drawn.

Take any other point M; and draw MC in the plane MN, MB in the plane MP, perpendicular to the common intersection AM. Since MB and AP are perpendicular to the same line AM, they are parallel to each other. For the same reason, MC is parallel to AN; therefore, (13. V.) the angle BMC=PAN; therefore it is indifferent whether the perpendiculars be drawn at the point M or at the point A; the included angle will be always the same.

In the second place, we must prove that, if the angle of the two planes increases or diminishes in a certain ratio, the angle PAN will increase or diminish in the same ratio.

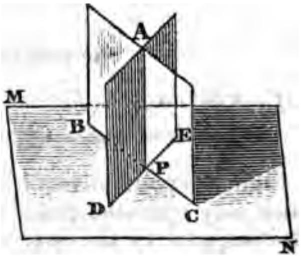
In the plane PAN, from the centre A and with any radius, describe the arc NDP; from the centre M and with an equal radius, describe the arc CEB; draw AD at pleasure: the two planes PAN, BMC, being perpendicular to the same straight line MA, will (9. V.) be parallel; therefore the intersections AD, ME, of these two planes with a third AMD, will be parallel; therefore (13. V.) the angle BME will be equal to PAD.

Let us for a moment call the angle, which is formed by the two planes MP, MN, a *wedge;* that granted, if the angle DAP were equal to DAN, it is evident that the wedge DAMP would be equal to the wedge DAMN; for the base PAD if placed upon its equal DAN would exactly coincide with it, the height AM would be always the same; therefore the two corners would coincide with each other. In like manner it may be shewn, that, if the angle DAP were contained a certain number of times exactly in the angle PAN, the wedge DAMP would be contained just as many times in the wedge PAMN. But from the ratio in whole numbers, the conclusion with regard to any ratio is legitimate, and was above demonstrated (17. II.) in a case altogether similar; therefore whatever be the ratio of the angle DAP to the angle PAN, the wedge DAMP will be in that same ratio with the wedge PAMN; therefore the angle NAP may be taken as the measure of the wedge PAMN, or of the angle which is formed by the two planes MAP, MAN.

Scholium. The same relation subsists between the angles which are formed by two planes, as between those which are formed by two straight lines. Thus when two planes mutually cross each other, the opposite or vertical angles are equal, and the adjacent angles are together equal to two right angles; therefore if one plane be perpendicular to another, the latter is also perpendicular to the former. In like manner, when two parallel planes are met by a third plane, the same equalities and the same properties appear, as when two parallel lines are met by a third line.

PROPOSITION XVIII. THEOREM.

If the line AP *is perpendicular to the plane* MN, *any plane* APB *drawn along* AP, *will be perpendicular to the plane* MN.



Let BC be the intersection of the planes AB, MN; in the plane MN, draw DE perpendicular to BP: then the line AP, being perpendicular to the plane MN, will be perpendicular to each of the two straight lines BC, DE; but the angle APD, formed by the two perpendiculars PA, PD at the common intersection BP, measures the angle of the two planes AB, MN; therefore (*Def.* 5.), since that angle is right, the two planes are perpendicular to each other.

Scholium. When three straight lines, such as AP, BP, DP, are perpendicular to each other, each of those lines is perpendicular to the plane of the other two, and the three planes are perpendicular to each other.

PROPOSITION XIX. THEOREM.

If the plane AB *is perpendicular to the plane* MN, *and if in the plane* AB *the line* PA *is perpendicular to the common intersection* PB, *then will* PA *be perpendicular to the plane* MN.

For, in the plane MN (see the preceding figure), draw PD perpendicular to PB; then, because the planes are perpendicular, the angle APD is right; therefore the line AP is perpendicular to the two straight lines PB, PD; therefore it is perpendicular to their plane MN.

Cor. If the plane AB is perpendicular to the plane MN, and if at a point P of the common intersection we erect a perpendicular to the plane MN, that perpendicular will be in the plane AB; for, if not, then, in the plane AB, we might draw AP perpendicular to PB the common intersection, and this AP, at the same time, would be perpendicular to the plane MN; therefore at the same point P there would be two perpendiculars to the plane MN, which is impossible.

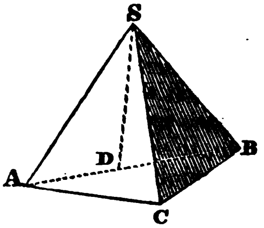
PROPOSITION XX. THEOREM.

If two planes AB, AD, *are perpendicular to a third* MN, *their common intersection* AP *will be perpendicular to that third plane.*

For, at the point P (see the preceding figure), erect a perpendicular to the plane MN; that perpendicular must be at once in the plane AB and in the plane AD (19. V. Cor.); therefore it is their common intersection AP.

PROPOSITION XXI. THEOREM.

If a solid angle is formed by three plane angles, the sum of any two of these angles will be greater than the third.



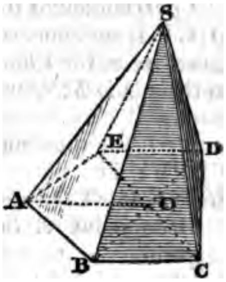
The proposition requires demonstration only when the plane angle, which is compared to the sum of the other two, is greater than either of them. Therefore suppose the solid angle S to be formed by three plane angles ASB, ASC, BSC, whereof the angle ASB is the greatest; we are to shew that $ASB<ASC+BSC$.

In the plane ASB make the angle $BSD=BSC$, draw the straight line ADB at pleasure; and having taken $SC=SD$, join AC, BC.

The two sides BS, SD are equal to the two BS, SC; the angle $BSD=BSC$; therefore the triangles BSD, BSC are equal; therefore $BD=BC$. But $AB<AC+BC$; taking BD from the one side, and from the other its equal BC, there remains $AD<AC$. The two sides AS, SD are equal to the two AS, SC; the third side AD is less than the third side AC; therefore (10. I.) the angle $ASD<ASC$. Adding $BSD=BSC$, we shall have $ASD+BSD$ or $ASB<ASC+BSC$.

PROPOSITION XXII. THEOREM.

The sum of the plane angles which form a solid angle, is always less than four right angles.



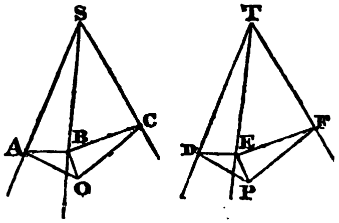
Cut the solid angle S by any plane ABCDE; from O, a point in that plane, draw to the several angles straight lines AO, OB, OC, OD, OE.

The sum of the angles of the triangles ASB, BSC, &c. formed about the vertex S, is equivalent to the sum of the angles of an equal number of triangles AOB, BOC, &c. formed about the point O. But at the point B the angles ABO, OBC, taken together, make the angle ABC (21. V.) less than the sum of the angles ABS, SBC; in the same manner, at the point C we have $BCO+OCD<BCS+SCD$; and so with all the angles of the polygon ABCDE: whence it follows, that the sum of all the angles at the bases of the triangles whose vertex is in O, is less than the sum of the angles at the bases of the triangles whose vertex is in S; hence to make up the deficiency, the sum of the angles formed about the point O, is greater than the sum of the angles about the point S. But the sum of the angles about the point O is equal to four right angles (5. I.); therefore the sum of the plane angles, which form the solid angle S, is less than four right angles.

Scholium. This demonstration is founded on the supposition that the solid angle is convex, or that the plane of no one surface produced can ever meet the solid angle; if it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude.

PROPOSITION XXIII. THEOREM.

If two solid angles are composed of three plane angles respectively equal to each other, the planes which contain the equal angles will be equally inclined to each other.



Let the angle ASC=DTF, the angle ASB=DTE, and the angle BSC=ETF; then will the inclination of the planes ASC, ASB, be equal to that of the planes DTF, DTE.

Having taken SB at pleasure, draw BO perpendicular to the plane ASC; from the point O, at which that perpendicular meets the plane, draw OA, OC perpendicular to SA, SC; join AB, BC; next take TE=SB; draw EP perpendicular to the plane DTF; from the point P draw PD, PF, perpendicular to TD, TF; lastly, join DE, EF.

The triangle SAB is right-angled at A, and the triangle TDE at D (6. V.); and since the angle we have ASB=DTE, SBA=TED. Likewise SB=TE; therefore the triangle SAB is equal to the triangle TDE; therefore SA=TD, and AB=DE. In like manner it may be shewn, that SC=TF, and BC=EF. That granted, the quadrilateral SAOC is equal to the quadrilateral TDPF: for, place the angle ASC upon its equal DTF; because SA=TD, and SC=TF, the point A will fall on D, and the point C on F; and at the same time, AO, which is perpendicular to SA, will fall on PD which is perpendicular to TD, and in like manner OC on PF; wherefore the point O will fall on the point P, and AO will be equal to DP. But the triangles AOB, DPE, are right-angled at O and P; the hypotenuse AB = DE, and the side AO = DP: hence (18. I.) those triangles are equal; hence the angle OAB=PDE. The angle OAB is the inclination of the two planes ASB, ASC; the angle PDE is that of the two planes DTE, DTF; hence those two inclinations are equal to each other.

It must, however, be observed, that the angle A of the right-angled triangle OAB is properly the inclination of the two planes ASB, ASC, only when the perpendicular BO falls on the same side of SA as SC falls; for if it fell on the other side, the angle of the two planes would be obtuse, and joined to the angle A of the triangle OAB it would make two right angles. But in the same case, the angle of the two planes TDE, TDF would also be obtuse, and joined to the angle D of the triangle DPE, it would make two right angles; and the angle A being thus always equal to the angle at D, it would follow in the same manner that the inclination of the two planes ASB, ASC, must be equal to that of the two planes TDE, TDF.

Scholium. If two solid angles are composed of three plane angles respectively equal to each other, and if at the same time the equal or homologous angles are *disposed in the same manner* in the two solid angles, these angles will be equal, and they will coincide when placed upon one another. We have already seen that the quadrilateral SAOC may be placed upon its equal TDPF; thus placing SA upon TD, SC falls upon TF, and the point O upon the point P. But, because the triangles AOB, DPE are equal, OB perpendicular to the plane ASC is equal to PE perpendicular to the plane TDF; besides, those perpendiculars lie in the same direction; therefore the point B will fall upon the point E, the line SB upon TE, and the two solid angles will wholly coincide.

This coincidence, however, takes place only when we suppose that the equal plane angles are *arranged in the same manner* in the two solid angles; for if they were *arranged in an inverse order*, or, what is the same, if the perpendiculars OB, PE, instead of lying in the same direction with regard to the planes ASC, DTF, lay in opposite directions, then it would be impossible to make these solid angles coincide with one another. It would not, however, on this account, be less true, as our Theorem states, that the planes containing the equal angles must still be equally inclined to each other; so that the two solid angles would be equal in all their constituent parts, without, however, admitting of superposition. This sort of equality, which is not absolute, or such as admits of superposition, deserves to be distinguished by a particular name: we shall call it *equality by symmetry.*

Thus those two solid angles, which are formed by three plane angles respectively equal to each other but disposed in an inverse order, will be called *angles equal by symmetry*, or simply *symmetrical angles.*

The same remark is applicable to solid angles, which are formed by more than three plane angles: thus a solid angle, formed by the plane angles A, B, C, D, E, and another solid angle, formed by the same angles in an inverse order A, E, D, C, B, may be such that the planes which contain the equal angles are equally inclined to each other. Those two solid angles, likewise equal without being capable of superposition, would be called *solid angles equal by symmetry*, or *symmetrical solid angles.*

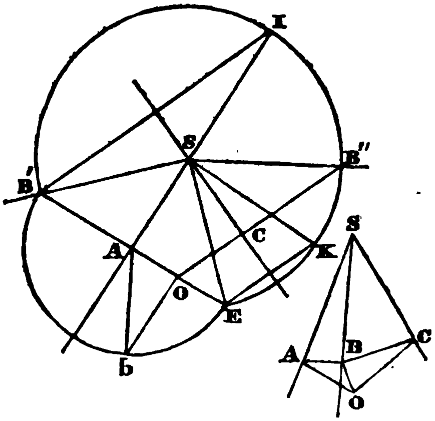
Among plane figures, equality by symmetry does not properly exist, all figures which might take this name being absolutely equal, or equal by superposition; the reason of which is, that a plane figure cannot be inverted, and the upper part taken indiscriminately for the under. This is not the case with solids; in which the third dimension may be taken in two different directions

PROPOSITION XXIV. PROBLEM.

The three angles which form a solid angle being given, to find by a plane construction the angle contained between two of these planes.

Let S be the proposed solid angle, in which the three plane angles ASB, ASC, BSC, are known, it is required to find

the angle contained by two of these planes, such as ASB, ASC.



Conceive the same construction to be made as in the preceding Theorem; the angle OAB would be the angle sought. It is required to find the same angle by a plane construction, or one performed on a plane.

On a plane, therefore, make the angles B′SA, ASC, B″SC equal to the angles BSA, ASC, BSC, in the solid figure; take B′S and B″S each equal to BS in the solid figure; from the points B′ and B″, at right angles to SA and SC draw B′A and B″C, which will intersect each other at the point O. From A as a centre, with the radius AB′, describe the semicircle B′*b*E; at the point O, erect O*b* perpendicular to B′E, and meeting the circumference in *b*; join A*b*: the angle EA*b* will be the required inclination of the two planes ASC, ASB in the solid angle.

All we have to prove is, that the triangle AO*b* of the plane figure is equal to the triangle AOB of the solid figure. Now the two triangles B′SA, BSA are right-angled at A; the angles at S are equal: hence the angles at B and B′ are also equal. But the hypotenuse SB′ is farther equal to the hypotenuse SB; hence these triangles are equal; therefore SA of the plane figure is equal to SA of the solid figure, and likewise AB′, or its equal A*b*, in the former to AB in the latter. In the same way, it might be shewn that SC is equal in both; hence it follows, that the quadrilateral SAOC must be equal in both, and consequently AO of the plane figure is equal to AO in the solid. Thus, in both figures the right-angled triangles AO*b*, AOB have each the hypotenuse and a side respectively equal; hence they are themselves equal; and the angle EAB, found by the plane construction, is equal to the inclination of the two planes SAB, SAC in the solid angle.

When the point O falls between A and B′ in the plane figure, the angle EAB becomes obtuse, and still measures the true inclination of the planes. It is for this reason that EA*b*, not OA*b*, has been employed to designate the required inclination, that so the same solution might suit every possible case.

Scholium. A question may arise, whether, if any three angles be assumed at pleasure, a solid angle can be formed with them.

Now, first, the sum of the three given angles must be less than four right angles, otherwise (22. V.) no solid angle can be formed; and farther, two of these angles B′SA, ASC being assumed at pleasure, the third CSB″ must be such that B″C, perpendicular to the side SC, shall meet the diameter B′E between its extremities B′ and E. Thus the limits to the magnitude of the angle CSB″ are such as would make the perpendicular B″C terminate in the points B′ and E. From these points, draw BI and EK at right angles to CS, and meeting the circumference described with the radius SB″ in I and K; the limits of the angle CSB″ will be CSI and CSK.

But in the isosceles triangle B′SI, since the line CS produced is perpendicular to the base B′I, we have the angle $CSI = CSB' = ASC + ASB'$. And in the isosceles triangle ESK, since the line SC is perpendicular to EK, we have the angle $CSK = CSE$. Also, by reason of the equal triangles ASE, ASB, we have the angle $ASE = ASB'$; hence CSE or $CSK = ASC - ASB'$.

Therefore the problem is always possible, when the third angle CSB″ is less than the sum of ASC, ASB′ the other two, and greater than their difference; conditions agreeing with Theorem 21.; according to which it was required that we should have $CSB'' < ASC + ASB'$, and also $ASC < CSB'' + ASB'$, or $CSB'' > ASC - ASB'$.

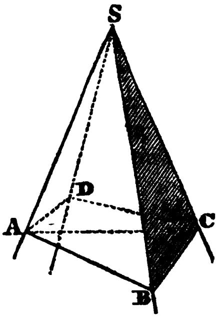
PROPOSITION XXV. PROBLEM.

Two of the three plane angles which form a solid angle, and also the inclination of their planes being given, to find the third plane angle.

Let ASC, ASB′, (see the last figure) be the two given plane angles; and suppose for a moment that CSB″ is the third angle required: then, employing the same construction as in the foregoing Problem, the angle included between the planes of the two first, the inclination of those planes, would be EA*b*. Now, as EA*b* can be determined by means of CSB″ the other two being given, so likewise may CSB″ be determined by means of EA*b*, which is just what the Problem requires.

Having taken SB′ at pleasure, upon SA let fall the indefinite perpendicular B′E; make the angle EA*b* equal to the inclination of the two given planes; from the point *b*, where the side A*b* meets the circle described from the centre A with the radius AB′, draw *b*O perpendicular to AE; from the point O, at right angles to SC draw the indefinite line OCB″; make SB″=SB′: the angle CSB″ will be the third plane angle required.

For, if a solid angle is formed with the three plane angles B′SA, ASC, CSB″, the inclination of the planes, in which are the given angles ASB′, ASC, will be equal to the given angle EAB.



Scholium. If a solid angle is *quadruple*, or formed by four plane angles ASB, BSC, CSD, DSA, a knowledge of all these angles is not enough for determining the mutual inclinations of their planes; for the same plane angles may serve to form a multitude of solid angles. But if one condition is added, if, for example, the inclination of the two planes ASB, BSC is given, then the solid angle is entirely determined, and the inclination of any other two of its planes may be found, as follows: Conceive a *triple* solid angle to be formed by the three plane angles ASB, BSC, ASC; the first two angles are given, as well as the inclination of their planes; the third angle ASC may therefore be determined, by the Problem we have just solved. Now, examining the triple solid angle formed by the plane angles ASC, ASD, DSC; those three angles are known; hence this solid angle is entirely determined. But the quadruple solid angle is formed by the junction of the two triple solid angles of which we have now been treating; and both these partial angles being known and determined, the whole angle will be known and determined likewise.

The inclination of the two planes ASD, DSC may be found immediately by means of the second partial solid angle. As for the inclination of the two planes BSC, CSD, to obtain this, the inclination of the two planes ASC, DSC must be found in the one partial angle, that of the two planes ASC, BSC in the other; the sum of these two inclinations will be the angle included between the planes BSC, DSC.

In the same manner, we should find that for determining a quintuple solid angle, not only the five plane angles which compose it must be known, but also two of the mutual inclinations of their planes; three in a sextuple solid angle; and so on.

BOOK VI.

POLYEDRONS.

Definitions.

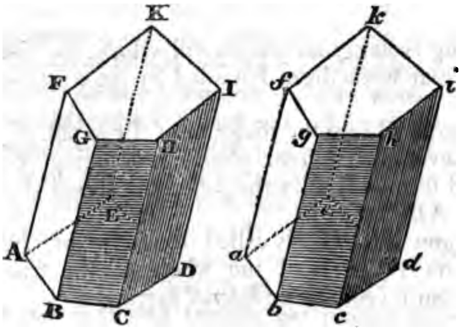
1. The name *solid polyedron*, or simply *polyedron*, is given to every solid terminated by planes or plane faces; which planes, it is evident, will themselves be terminated by straight lines. In particular, the solid which has four faces is named a *tetraedron;* that which has six, a *hexaedron;* that which has eight, an *octaedron;* that which has twelve, a *dodecaedron;* that which has twenty, an *iscocaedron;* and so on.

The tetraedron is the simplest of all polyedrons; because at least three planes are required to form a solid angle, and these three planes leave a void, which cannot be closed without at least one other plane.

II. The common intersection of two adjacent faces of a polyedron is called the *side* or *edge* of the polyedron.

III. A *regular polyedron* is one whose faces are all equal regular polygons, and whose solid angles are all equal to each other. There are five such polyedrons. (See the Appendix to Books VI. and VII.)

IV. The *prism* is a solid bounded by several plane parallelograms, which are terminated at both ends by two plane polygons equal and parallel.



To construct this solid, let ABCDE be any polygon; then if in a plane parallel to ABC, the lines FG, GH, HI, &c. be drawn equal and parallel to the sides AB, BC, CD, &c. thus forming the polygon FGHIK equal to ABCDE; if in the next place, the vertices of the angles in the one plane be joined with the homologous vertices in the other, by straight lines AF, BG, CH, &c., the faces ABGF, BCHG, &c. will be parallelograms, and ABCDEFGHIK, the solid so formed, will be a prism.

V. The equal and parallel polygons ABCDE, FGHIK are called the *bases of the prism;* the plane parallelograms taken together constitute the *lateral* or *convex surface of the prism;* the equal straight lines AF, BG, CH, &c. are called the *sides of the prism.*

VI. The *altitude of a prism* is the distance of its two bases, or the perpendicular drawn from a point in the upper base to the plane of the lower base.

VII. A *prism* is *right* when the sides AF, BG, CH, &c. are perpendicular to the planes of the bases; and then each of them is equal to the altitude of the prism. In any other case the prism is *oblique,* and the altitude less than the side.

VIII. A *prism* is *triangular, quadrangular, pentagonal, hexagonal,* &c. when the base is a triangle, a quadrilateral, a pentagon, a hexagon, &c.

IX. A prism whose base is a parallelogram has all its faces parallelograms; it is named a *parallelepipedon.* (See the diagram of Prop. 4. VI.)

The *parallelepipedon* is *rectangular* when all its faces are rectangles.

X. Among rectangular parallelepipedons, we distinguish the *cube,* or regular hexaedron, bounded by six equal squares.

XI. A *pyramid* (see the diagram of Prop. 22. V.) is the solid formed by several triangular planes proceeding from the same point S, and terminating in the different sides of the same polygonal plane ABCDE.

The polygon ABCDE is called the *base* of the pyramid, the point S its *vertex;* and the whole of the triangles ASB, BSC, &c. form its *convex* or *lateral surface.*

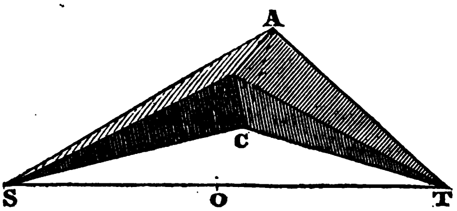
XII. The *altitude* of a pyramid is the perpendicular let fall from the vertex upon the plane of the base, produced if necessary.

XIII. A pyramid is *triangular*, *quadrangular*, &c. according as its base is a triangle, a quadrilateral, &c.

XIV. A pyramid is *regular*, when its base is a regular polygon, and when, at the same time, the perpendicular let fall from the vertex to the plane of the base passes through the centre of this base. That perpendicular is then called the *axis* of the pyramid.

XV. The *diagonal* is the straight line joining the vertices of two solid angles which are not adjacent to each other.

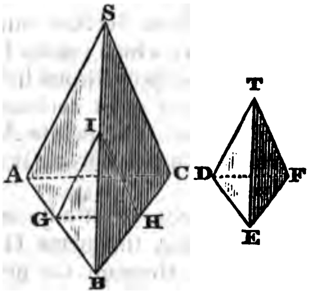
XVI. We shall give the name, *symmetrical polyedrons*, to any two polyedrons which having a common base, are constructed similarly, the one above this base, the other beneath it, and so that the vertices of their corresponding solid angles are situated at equal distances from the plane of the base, and in the same straight line perpendicular to that plane.



If the straight line ST, for example, is perpendicular to the plane ABC, and also bisected at the point O, where it meets this plane, the two pyramids SABC, TABC, which have the common base ABC, will be two symmetrical polyedrons.

XVII. Two *triangular pyramids* are *similar*, when two faces in each are respectively similarly placed, and equally inclined to each other.

Thus, supposing the angles ABC=DEF, BAC=EDF, ABS =DET, BAS=EDT, if the inclination of the planes ABS, ABC is likewise equal to that of their homologous planes DTE, DEF, the pyramids SABC, TDEF will be similar.



XVIII. If a triangle is formed by joining the vertices of three angles taken on one face or on the base of a polyedron, then the vertices of the different solid angles of the polyedron, which are situated without this base may be conceived as being the vertices of so many triangular pyramids having the triangle just described for a common base; and each of those pyramids will determine the position of a solid angle of the polyedron with reference to the base. Now,

Two *polyedrons* are *similar*, when having similar bases, the vertices of their corresponding solid angles lying without those bases are determined by triangular pyramids which are similar each to each.

XIX. By the *vertices* of a polyedron, we mean the points situated at the vertices of its different solid angles.

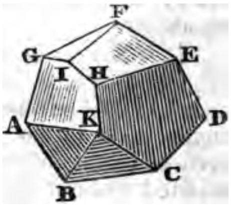
Note. The only polyedrons we intend at present to treat of, are polyedrons with salient angles, or *convex* polyedrons. They are such that their surface cannot be intersected by a straight line in more than two points. In polyedrons of this kind, the plane of any face, when produced, can in no case cut the solid; the polyedron therefore cannot be in part above the plane of any face, and in part below it; it must lie wholly on the same side of this plane.

PROPOSITION I. THEOREM.

Two polyedrons cannot have the same vertices and in the same number without coinciding.

For, suppose one polyedron to be already constructed: if a second is to be formed, having the same vertices and in the same number, the planes of the latter must either not all pass through the same points with the planes of the former, or the two polyedrons will not differ from each other. But if those planes of the latter do not all pass through the same points with the planes of the former, some of them must cut the first polyedron; one or more of whose vertices must therefore lie above them, one or more below; which cannot be the case with a convex polyedron: hence if two polyedrons have the same vertices and in the same number, they must necessarily coincide with each other.

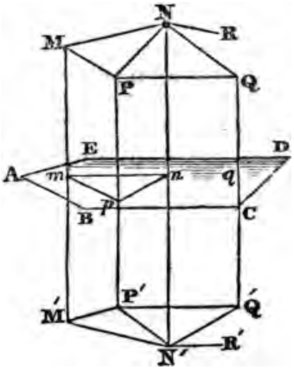
Scholium. The points A, B, C, K, &c. which are to be the vertices of a polyedron, being given, it is easy to describe the polyedron.



First choose three adjacent points D, E, H, such that the plane DEH shall pass, if need be, through the new points K, C, but leaving all the rest on the same side, all above the plane or all below it; the plane DEH or DEHKC, thus determined, will be one face of the solid. Along EH one of its sides, extend a plane, which turn round upon EH as an axis till it meet a new vertex F, or several at once as F, I; it will give a second face FEH or FEHI. Continue the same process, making planes to pass through the sides successively determined, till the solid is bounded on all quarters: this solid will be the polyedron required, since there cannot be two which have the same vertices.

PROPOSITION II. THEOREM.

In two symmetrical polygons, the homologous faces are respectively equal, and the inclination of two adjacent faces in one of those solids is equal to the inclination of the two homologous faces in the other.



Let ABCDE be the common base of the two polyedrons; M and N the vertices of any two solid angles in the one, M′ and N′ the homologous vertices of the other; then (*Def.* 16.) the straight lines MM′, NN′, must be perpendicular to the plane ABC, and be divided into two equal parts at the points *m* and *n*, where they meet it. Now we are to shew that MN is equal to M′N′.

For, if the trapezium *m*M′N′*n* be made to revolve about *mn* till the plane of it falls upon the plane *m*MN*n*; by reason of the right angles at *m* and *n*, the side *m*M′ will fall on its equal *m*M, and *n*N′ upon *n*N; hence the trapeziums will coincide, and we shall have MN=M′N′.

Let P be a third vertex of the upper polyedron, and P′ its homologous vertex in the other; we shall, as before, have MP=MP′, and NP=NP′; hence *the triangle* MNP, *which joins any three vertices of the upper polyedron, is equal to the triangle* M′N′P′, *which joins the three corresponding vertices of the other polyedron.*

If, among those triangles, we confine our attention to such as are formed at the surface of the polyedrons, we may already conclude that the surfaces of the two polyedrons are each composed of the same number of triangles respectively equal in both.

It is now to be shewn, that, if any of those triangles lie on the same plane in the upper surface, and form one and the same polygonal face, the corresponding triangles will lie on the same plane in the under surface, and there form one equal polygonal face.

To prove this, let MPN, NPQ, be two adjacent triangles supposed to lie on the same plane; and let M′P′N′, N′P′Q′, be their corresponding triangles. The angle MNP=M′N′P′, the angle PNQ=P′N′Q′; and if MQ and M′Q′ were joined, the triangle MNQ would be equal to M′N′Q′; hence we should have the angle MNQ=M′N′Q′. But since MPNQ is one single plane, the angle MNQ=MNP+PNQ; hence we shall likewise have M′N′Q′=M′N′P′+P′N′Q′. Now, if the three planes M′N′P′, P′N′Q′, M′N′Q′ were not all in one plane, those three planes would form a solid angle, and (20. V.) we should have the angle $M'N'Q' \angle M'N'P'+P'N'Q'$; which conclusion not being true, the two triangles M′N′P′, P′N′Q′ are in one and the same plane.

Hence each face, whether triangular or polygonal, in the one polyedron, corresponds to an equal face in the other polyedron, and thus the two polyedrons are each included under the same number of planes respectively equal in both.

We have still to shew, that the inclination of any two adjacent faces in the one polyedron is equal to the inclination of the two corresponding faces in the other.

Let MPN, NPQ be two triangles formed on the common edge NP, in the planes of two adjacent faces; let M′P′N′, N′P′Q′ correspond to them: conceive a solid angle to be formed at N, by the three plane angles MNQ, MNP, PNQ; and another at N′, by the three M′N′Q′, M′N′P′, P′N′Q′. Now it has been shewn already, that those plane angles are respectively equal; hence the inclination of the two planes MNP, PNQ is equal (22. V.) to that of their corresponding planes M′N′P′, P′N′Q′.

Therefore, in symmetrical polyedrons, the faces are equal each to each; and the planes of any two adjacent faces, in the one solid have the same inclination as the planes of the two corresponding faces in the other solid.

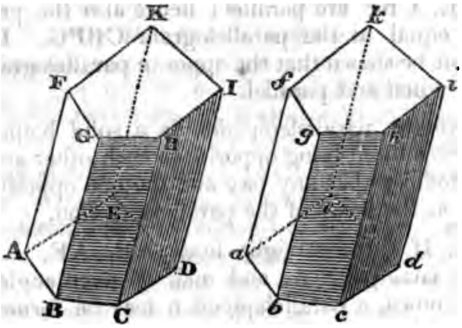
Scholium. It may be observed, that *the solid angles of the one polyedron are symmetrical with the solid angles of the other;* for as the solid angle N is formed by the planes MNP, PNQ, QNR, &c., so its corresponding angle N is formed by the planes M′N′P′, P′N′Q′, Q′N′R′, &c. The latter appear to be arranged in the same order as the former; but since the two solid angles are in an inverse position with regard to each other, the real arrangement of the planes which form the solid angle N′, must be the reverse of the arrangement which occurs in the corresponding angle N. Farther, in both solids, the inclinations of the consecutive planes are respectively the same; hence those solid angles are symmetrical each with the other. (See the scholium of Prop. 23. V.)

The observation we have just made proves, moreover, that *no polyedron can have more than one polyedron symmetrical with it.* For if upon a second base, a new polyedron were constructed symmetrical with the given one, the solid angles of this new polyedron would still be symmetrical with those of the given polyedron; hence they would be equal to those of the symmetrical polyedron constructed on the first base. Besides, the homologous faces would still be equal: hence those two symmetrical polyedrons constructed on the first and on the second base, would have their faces equal and their solid angles equal; hence they would coincide if applied to each other; hence they would form one and the same polyedron.

PROPOSITION III. THEOREM.

Two prisms are equal when a solid angle in each is contained by three planes which are similarly placed, and respectively equal in both.

Let the base ABCDE be equal to the base *abcde*, the parallelogram ABGF equal to the parallelogram *abgf*, and the parallelogram BCHG equal to *bchg*; then will the prism ABCI be equal to the prism *abci*.

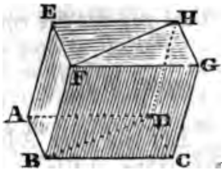


For, lay the base ABCDE upon its equal *abcde;* these two bases will coincide. But the three plane angles, which form the solid angle B, are respectively equal to the three plane angles, which form the solid angle *b*, namely, ABC=*abc*, ABG=*abg*, and GBC=*gbc;* they are also similarly situated: hence the solid angles B and *b* are equal, and therefore the side BG will fall on its equal *bg*. It is likewise evident, that by reason of the equal parallelograms ABGF, *abgf*, the side GF will fall on its equal *gf*, and in the same manner GH on *gh;* hence the upper base FGHIK will exactly coincide with its equal *fghik*, and the two solids will be identical (1. VI.), since their vertices are the same.

Cor. Two right prisms, which have equal bases and equal altitudes, are equal. For, since the side AB is equal to *ab*, and the altitude BG to *bg*, the rectangle ABGF will be equal to *abgf;* so also will the rectangle BGHC be to *bghc;* and thus the three planes, which form the solid angle B, will be equal to the three which form the solid angle *b*. Hence the two prisms are equal.

PROPOSITION IV. THEOREM.

In every parallelepipedon the opposite planes are equal and parallel.



By the definition of this solid, the bases ABCD, EFGH are equal parallelograms, and their sides are parallel: it remains only to shew, that the same is true of any two opposite lateral faces, such as AEHD, BFGC. Now AD is equal and parallel to BC, because the figure ABCD is a parallelogram; for a like reason, AE is parallel to BF: hence (13. V.) the angle DAE is equal to the angle CBF, and the planes DAE, CBF are parallel; hence also the parallelogram DAEH is equal to the parallelogram CBFG. In the same way, it might be shewn that the opposite parallelograms ABFE, DCGH are equal and parallel.

Cor. Since the parallelepipedon is a solid bounded by six planes, whereof those lying opposite to each other are equal and parallel, it follows that any face and the one opposite to it may be assumed as the bases of the parallelepipedon.

Scholium. If three straight lines AB, AE, AD, passing through the same point A, and making given angles with each other, are known, a parallelepipedon may be formed on those lines. For this purpose, a plane must be extended through the extremity of each line, and parallel to the plane of the other two; that is, through the point B a plane parallel to DAE, through D a plane parallel to BAE, and through E a plane parallel to BAD. The mutual intersections of those planes will form the parallelepipedon required.

PROPOSITION V. THEOREM.

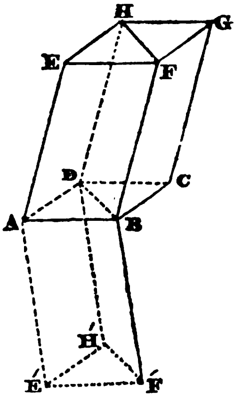
In every parallelepipedon, the opposite solid angles are symmetrical; and the diagonals drawn through the vertices of those angles bisect each other.

First. Compare the solid angle A (see preceding figure) with its opposite one G. The angle EAB, equal to EFB, is also equal to HGC; the angle DAE=DHE=CGF; and the angle DAB=DCB=HGF; therefore the three plane angles, which form the solid angle A, are equal to the three which form the solid angle G, each to each. It is easy, moreover, to see that their arrangement in the one is different from their arrangement in the other: hence (23. V.) the two solid angles A and G are symmetrical.

Secondly. Imagine two diagonals EC, AG to be drawn both through opposite vertices: since AE is equal and parallel to CG, the figure AEGC is a parallelogram; hence the diagonals EC, AG will mutually bisect each other. In the same manner, we could shew that the diagonal EC and another DF bisect each other; hence the four diagonals will mutually bisect each other, in a point which may be regarded as the centre of the parallelepipedon.

PROPOSITION VI. THEOREM.

If the plane BDHF *pass through two opposite and parallel edges* BF, DH, *it will divide the parallelepipedon* AG *into two triangular prisms* ABDHEF, GHFBCD, *which will be symmetrical with each other.*



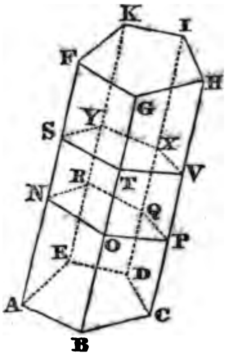
In the first place, those solids are evidently prisms; for the triangles ABD, EFH, having their sides equal and parallel, are equal; also the lateral faces ABFE, ADHE, BDHF are parallelograms; hence the solid ABDHEF is a prism: so likewise is GHFBCD. We are now to shew that those prisms are symmetrical.

On the base ABD, construct the prism ABDE′F′H′ such that it be symmetrical with the prism ABDEFH. According to what has been already proved (2. VI.), the plane ABF′E′ must be equal to ABFE, and the plane ADH′E′ to ADHE: but comparing the prism GHFBCD with the prism

ABDH′E′F′, we find the base GHF=ABD; the parallelogram GHDC, which is equal to ABFE, also equal to ABF′E′; and the parallelogram GFBC, which is equal to ADHE, also equal to ADH′E′: therefore the three plane planes which form the solid angle G in the prism GHFBCD are equal to the three planes which form the solid angle A in the prism ABDH′E′F′, each to each; and they are similarly arranged: hence those two prisms are equal, and might be made to coincide. But one of them ABDH′E′F′ is symmetrical with the prism ABHEF; hence the other GHFBCD is also symmetrical with that prism.

PROPOSITION VII. LEMMA.

In every prism ABCI, *the sections* NOPQR, STVXY, *formed by parallel planes, are equal polygons.*

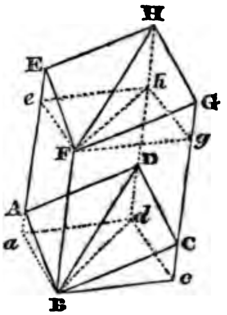


For the sides ST, NO, are parallel, being the intersections of two parallel planes with a third plane ABGF; those same sides ST, NO, are included between the parallels NS, OT, which are sides of the prism: hence NO is equal to ST. For like reasons, the sides OP, PQ, QR, &c. of the section NOPQR, are respectively equal to the sides TV, VX, XY, &c. of the section STVXY. And since the equal sides are at the same time parallel, it follows that the angles NOP, OPQ, &c. of the first section are respectively equal to the angles STV, TVX of the second. Hence the two sections NOPQR, STVXY are equal polygons.

Cor. Every section in a prism, if drawn parallel to the base, is also equal to that base.

PROPOSITION VIII. THEOREM.

The two symmetrical triangular prisms ABDHEF, BCDFGH, *into which the parallelepipedon* AG *may be decomposed, are equivalent.*



Through the vertices B and F, draw the planes B*adc*, F*ehg* at right angles to the side BF, and meeting AE, DH, CG the three other sides of the parallelepipedon, in the points *a*, *d*, *c* towards one direction, in *e*, *h*, *g* towards the other: the sections B*adc*, F*ehg* will be equal parallelograms. They are equal (7. VI.), because they are formed by planes perpendicular to the same straight line, and consequently parallel; they are parallelograms, because *a*B, *dc*, two opposite sides of the same section, are formed by the meeting of one plane with two parallel planes ABFE, DCGH.

For a like reason, the figure B*ae*F is a parallelogram; so also are BF*gc*, *cdhg*, *adhe*, the other lateral faces of the solid B*adc*F*ehg*; hence that solid is a prism (*Def.* 4. VI.); and that prism is right, because the side BF is perpendicular to its base.

This being proved, if the right prism B*h* is divided, by the plane BFHD, into two right triangular prisms *a*B*de*F*h*, B*dc*F*hg*; we are now to shew that the oblique triangular prism ABDEFH will be equivalent to the right triangular prism *a*B*de*F*h*. And since those two prisms have a part ABD*he*F in common, it will only be requisite to prove that the remaining parts, namely, the solids B*a*AD*d*, F*e*EH*h* are equivalent.

Now, by reason of the parallelograms ABFE, *a*BF*e*, the sides AE, *ae*, being equal to their parallel BF, are equal to each other; and taking away the common part A*e*, there remains A*a*=E*e*. In the same manner we could prove D*d*=H*h*.

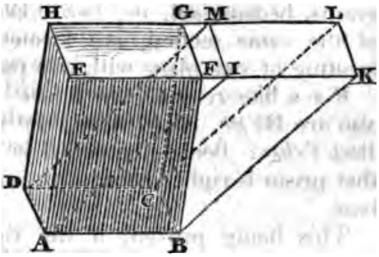
Next, to bring about the superposition of the two solids B*a*AD*d*, F*e*EH*h*, let us place the base F*eh* on its equal B*ad*: the point *e* falling on *a*, and the point *h* on *d*, the sides *e*E, *h*H will fall on their equals *a*A, *d*D, because they are perpendicular to the same plane B*ad*. Hence the two solids in question will coincide exactly with each other; hence the oblique prism BADFEH is equivalent to the right one B*ad*F*eh*

In the same manner might the oblique prism BDCFHG be proved equivalent to the right prism B*dc*F*hg*. But (3. VI. Cor.) the two right prisms B*ad*F*eh*, B*dc*F*hg* are equal, since they have the same altitude BF, and since their bases B*ad*, B*dc* are halves of the same parallelogram. Hence the two triangular prisms BADFEH, BDCFHG, being equivalent to the equal oblique prisms, are equivalent to each other.

Cor. Every triangular prism ABDHEF is half of the parallelepipedon AG described on the same solid angle A, with the same edges AB, AD, AE.

PROPOSITION IX. THEOREM.

If two parallelepipedons AG, AL *have a common base* ABCD, *and if their upper bases* EFGH, IKLM *lie in the same plane and between the same parallels* EK, HL, *those two parallelepipedons will be equivalent.*



There may be three cases, according as EI is greater, less than, or equal to EF; but the demonstration is the same for all. In the first place, then, we shall shew that the triangular prism AEIDHM is equal to the triangular prism BFKCGL.

Since AE is parallel to BF, and HE to GF, the angle AEI=BFK, HEI=GFK, and HEA=GFB. Of these six angles the first three form the solid angle E, the last three the solid angle F; therefore, the plane angles being respectively equal, and similarly arranged, the solid angles F and E must be equal. Now, if the prism AEM is laid on the prism BFL, the base AEI being placed on the base BFK will coincide with it because they are equal; and since the solid angle E is equal to the solid angle F, the side EH will fall on its equal FG: and nothing more is required to prove the coincidence of the two prisms throughout their whole extent, for (3. VI.) the base AEI and the edge EH determine the prism AEM, as the base BFK and the edge FG determine the prism BFL; hence these prisms are equal.

But if the prism AEM is taken away from the solid AL, there will remain the parallelepipedon AIL; and if the prism BFL is taken away from the same solid, there will remain the parallelepipedon AEG; hence those two parallelepipedons AIL, AEG are equivalent.

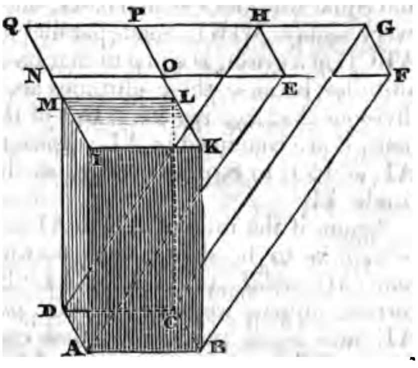
PROPOSITION X. THEOREM.

Two parallelepipedons having the same base and the same altitude are equivalent.

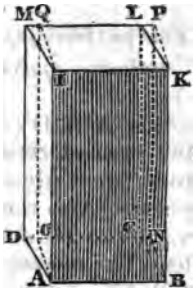
Let ABCD (see the next figure) be the common base of the two parallelepipedons AG, AL: since they have the same altitude, their upper bases EFGH, IKLM will be in the same plane. Also the sides EF and AB will be equal and parallel, as well as IK and AB; hence EF is equal and parallel to IK; for a like reason, GF is equal and parallel to LK. Let the sides EF, HG be produced, and likewise LK, IM, till by their intersections they form the parallelogram NOPQ; this parallelogram will evidently be equal to either of the bases EFGH, IKLM. Now if a third parallelepipedon be conceived, having for its lower base the same ABCD, and NOPQ for its upper, this third parallelepipedon will (9. VI.) be equal to the parallelepipedon AG, since with the same lower base, their upper bases lie in the same plane and between the same parallels GQ, FN. For the same reason, this third parallelepipedon will also be equivalent to the parallelepipedon AL; hence the two parallelepipedons AG, AL, which have the same base and the same altitude, are equivalent.

PROPOSITION XI. THEOREM.

Any parallelepipedon may be changed into an equivalent rectangular parallelepipedon having the same altitude and an equivalent base.



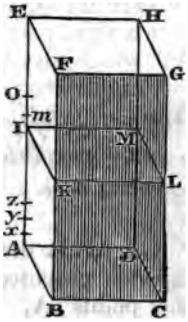
Let AG be the parallelepipedon proposed. From the points A, B, C, D, draw AI, BK, CL, DM, perpendicular to the plane of the base; you will thus form the parallelepipedon AL equivalent to AG, and having its lateral faces AK, BL, &c. rectangular. Hence if the base ABCD is a rectangle, AL will be the rectangular parallelepipedon equivalent to AG, the parallelepipedon proposed. But if ABCD



is not a rectangle, draw AO and BN perpendicular to CD, and OQ and NP perpendicular to the base; you will then have the solid ABNOIPQ, which will be a rectangular parallelepipedon: for by construction, the base ABNO and its opposite IKPQ are rectangles; so also are the lateral faces, the edges AI, OQ, &c. being perpendicular to the plane of the base; hence the solid AP is a rectangular parallelepipedon. But the two parallelepipedons AP, AL may be conceived as having the same base ABKI and the same altitude AO: hence the parallelepipedon AG, which was at first changed into an equivalent parallelepipedon AL, is again changed into an equivalent rectangular parallelepipedon AP, having the same altitude AI, and a base ABNO equivalent to the base ABCD.

PROPOSITION XII. THEOREM.

Two rectangular parallelepipedons AG, AL, *which have the same base* ABCD, *are to each other as their altitudes* AE, AI.

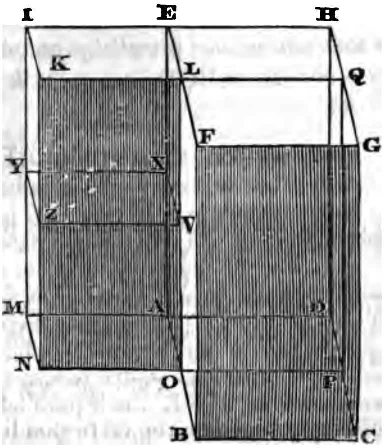


First, suppose the altitudes AE, AI, to be to each other as two whole numbers, as 15 is to 8, for example. Divide AE into 15 equal parts; whereof AI will contain 8; and through x, y, z, &c. the points of division, draw planes parallel to the base. These planes will cut the solid AG into 15 partial parallelepipedons, all equal to each other, because having equal bases and equal altitudes,—equal bases, since (7. VI.) every section MIKL, made parallel to the base ABCD of a prism, is equal to that base,—equal altitudes because these altitudes are the same divisions Ax, xy, xy, &c. But of those 15 equal parallelepipedons, 8 are contained in AL; hence the solid AG is to the solid AL as 15 is to 8, or generally, as the altitude AE is to the altitude AI.

Again, if the ratio of AE to AI cannot be expressed in numbers, it is to be shewn, that notwithstanding, we shall have *solid.* AG : *solid.* AL : : AE : AI. For, if this proportion is not correct, suppose we have *sol.* AG : *sol.* AL : : AE : AO. Divide AE into equal parts, such that each shall be less than OI; there will be at least one point of division m between O and I. Let P be the parallelepipedon, whose base is ABCD, and altitude A*m*; since the altitudes AE, A*m* are to each other as two whole numbers, we shall have *sol.* AG : P : : AE : A*m*. But by hypothesis, we have *sol.* AG : *sol.* AL : : AE : AO; therefore *sol.* AL : P : : AO : A*m*. But AO is greater than A*m*; hence if the proportion is correct, the solid AL must be greater than P. On the contrary, however, it is less: hence the fourth term of this proportion *sol.* AG : *sol.* AL : : AE : x, cannot possibly be a line greater than AI. By the same mode of reasoning, it might be shewn that the fourth term cannot be less than AI; therefore it is equal to AI: hence rectangular parallelepipedons having the same base are to each other as their altitudes.

PROPOSITION XIII. THEOREM.

Two rectangular parallelepipedons AG, AK, *having the same altitude* AE, *are to each other as their bases* BBCD, AMNO.



Having placed the two solids by the side of each other, as the figure represents, produce the plane ONKL till it meets the plane DCGH in PQ; you will thus have a third parallelepipedon AQ, which may be compared with each of the parallelepipedons AG, AK. The two solids AG, AQ, having the same base AEHD, are to each other as their altitudes AB, AO; in like manner, the two solids AQ, AK, having the same base AOLE, are to each other as their altitudes AD, AM. Hence we have the two proportions,

sol. AG : *sol.* AQ : : AB : AO,
sol. AQ : *sol.* AK : : AD : AM.

Multiply together the corresponding terms of those proportions, omitting in the result the common multiplier *sol.* AQ; we shall have

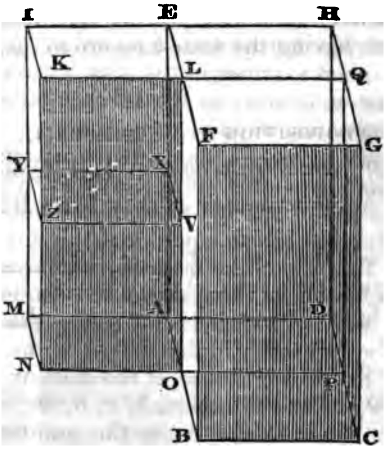
sol. AG : *sol.* AK : : AB×AD : AO×AM.

But AB×AD represents the base ABCD; and AO×AM represents the base AMNO: hence two rectangular parallelepipedons of the same altitude are to each other as their bases.

PROPOSITION XIV. THEOREM.

Any two rectangular parallelepipedons are to each other as the products of their bases by their altitudes, that is to say, as the products of their three dimensions.

For, having placed the two solids AG, AZ, so that their surfaces have the common angle BAE, produce the interior planes necessary for completing the third parallelepipedon AK, having the same altitude with the parallelepipedon AG. By the last Proposition, we shall have



sol. AG : *sol.* AK : : ABCD : AMNO.

But the two parallelepipedons AK, AZ having the same base AMNO, are to each other as their altitudes AE, AX; hence we have

sol. AK : *sol.* AZ : : AE : AX.

Multiply together the corresponding terms of those proportions, omitting in the result the common multiplier *sol.* AK; we shall have

sol. AG : *sol.* AZ : : ABCD × AE : AMNO × AX.

Instead of the bases ABCD and AMNO, put AB × AD and AO × AM; it will give

sol. AG : *sol.* AZ : : AB × AD × AE : AO × AM × AX.

Hence any two rectangular parallelepipedons are to each other, &c.

Scholium. We are consequently authorised to assume, as the measure of a rectangular parallelepipedon, the product of its base by its altitude, in other words, the product of its three dimensions.

In order to comprehend the nature of this measurement, it is necessary to reflect, that by the product of two or more lines is always meant the product of the numbers which represent them, those numbers themselves being determined by the linear unit, which may be assumed at will. Upon this principle, the product of the three dimensions of a parallelepipedon is a number, which signifies nothing of itself, and would be different if a different linear unit had been assumed. But if the three dimensions of another parallelepipedon are valued according to the same linear unit, and multiplied together in the same manner, the two products will be to each other as the solids, and will serve to express their relative magnitude.

The magnitude of a solid, its volume or extent, form what is called its *solidity*; and this word is exclusively employed to designate the measure of a solid: thus we say the solidity of a rectangular parallelepipedon is equal to the product of its base by its altitude, or to the product of its three dimensions.

As the cube has all its three dimensions equal, if the side is 1, the solidity will be $1 \times 1 \times 1 = 1$; if the side is 2, the solidity will be $2 \times 2 \times 2 = 8$; if the side is 3, the solidity will be $3 \times 3 \times 3 = 27$; and so on: hence, if the sides of a series of cubes are to each other as the numbers, 1, 2, 3, &c. the cubes themselves or their solidities will be as the numbers 1, 8, 27, &c. Hence it is, that in arithmetic, the *cube* of a number is the name given to the product which results from three factors each equal to this number.

If it were proposed to find a cube double of a given cube, the side of the required cube would have to be to that of the given one, as the cube-root of 2 is to unity. Now, by a geometrical construction, it is easy to find the square-root of 2; but the cube-root of it cannot be so found, at least not by the simple operations of elementary geometry, which consist in employing nothing but straight lines, two points of which are known, and circles whose centres and radii are determined.

Owing to this difficulty the problem of the *duplication of the cube* became celebrated among the ancient geometers, as well as that of the *trisection of an angle*, which is nearly of the same species. The solutions of which such problems are susceptible have, however, long since been discovered; and though less simple than the constructions of elementary geometry, they are not, on that account, less rigorous or less satisfactory.

PROPOSITION XV. THEOREM.

The solidity of a parallelepipedon, and generally of any prism, is equal to the product of its base by its altitude.

For, in the first place, any parallelepipedon (11. VI.) is equivalent to a rectangular parallelepipedon, having the same altitude and an equivalent base. Now the solidity of the latter is equal to its base multiplied by its height; hence the solidity of the former is, in like manner, equal to the product of its base by its altitude.

In the second place, any rectangular prism (8. VI.) is half of the parallelepipedon so constructed as to have the same altitude and a double base. But the solidity of the latter is equal to its base multiplied by its altitude; hence that of a triangular prism is also equal to the product of its base (half that of the parallelepipedon) multiplied into its altitude.

In the third place, any prism may be divided into as many triangular prisms of the same altitude, as there are triangles capable of being formed in the polygon which constitutes its base. But the solidity of each triangular prism is equal to its base multiplied by its altitude; and since the altitude is the same for all, it follows that the sum of all the partial prisms must be equal to the sum of all the partial triangles, which constitute their bases, multiplied by the common altitude.

Hence the solidity of any polygonal prism is equal to the product of its base by its altitude.

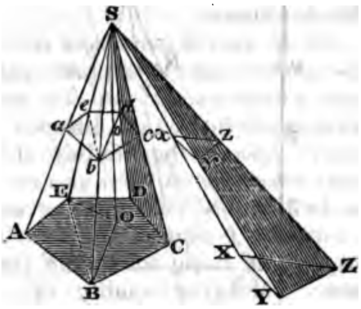
Cor. Comparing two prisms, which have the same altitude, the products of their bases by their altitudes will be as the bases simply; hence *two prisms of the same altitude are to each other as their bases.* For a like reason, *two prisms of the same base are to each other as their altitudes.*

PROPOSITION XVI. LEMMA.

If a pyramid SABCDE *is cut by a plane* abd *parallel to its base,*

First, *The sides* SA, SB, SC, *&c. and the altitude* SO, *will be divided proportionally in a, b, c,* &c. *and o.*

Secondly, *The section* abcde *will be a polygon similar to the base* ABCDE.



First. Since the planes ABC, *abc* are parallel, their intersections AB, *ab*, by a third plane AB will also be parallel (10. V.); hence the triangles SAB, S*ab* are similar, and we have SA : S*a* : : SB : S*b*; we might also have SB : S*b* : : SC : S*c*; and so on. Hence all the sides SA, SB, SC, &c. are cut proportionally in *a*, *b*, *c*, &c. The altitude SO is likewise cut in the same proportion, at the point *o*; for BO and B*o* are parallel, therefore we have SO : S*o* : : SB : S*b*.

Secondly. Since *ab* is parallel to AB, *bc* to BC, *cd* to CD, &c. the angle *abc* is equal to ABC, the angle *bcd* to BCD, and so on. Also, by reason of the similar triangles SAB, S*ab*, we have AB : *ab* : : SB : S*b*; and by reason of the similar triangles SBC, S*bc*, we have SB : S*b* : : BC : *bc*; hence AB : *ab* : : BC : *bc*; we might likewise have BC : *bc* : : CD : *cd*, and so on. Hence the polygons ABCDE, *abcde* have their angles respectively equal and their homologous sides proportional; hence they are similar.

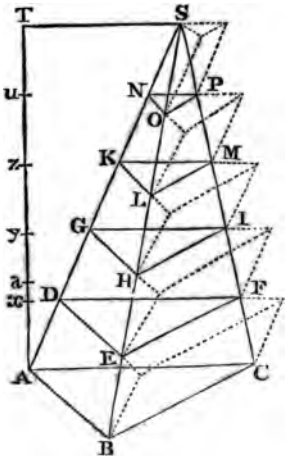
Cor. 1. Let SABCDE, SXYZ be two pyramids, having a common vertex and the same altitude, or having their bases situated in the same plane; if those pyramids are cut by one plane parallel to the plane of their bases, and the sections *abcde*, *xyz* result from it, *then will the sections* abcde, xyz *be to each other as the bases* ABCD, XYZ.

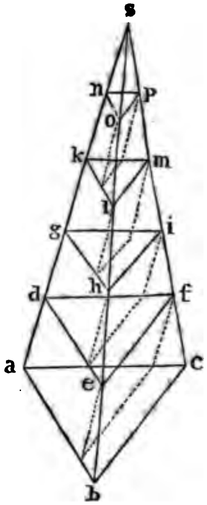
For, the polygons ABCD, *abcde* being similar, their surfaces are as the squares of the homologous sides AB, *ab*; but AB : *ab* : : SA : S*a*; hence ABCDE : *abcde* : : $\mathrm{SA}^2 : \mathrm{S}a^2$. For the same reason, XYZ : *xyz* : $\mathrm{SX}^2 : \mathrm{S}x^2$. But since *abcxyz* is all in one plane, we have likewise SA : S*a* : : SX : S*x*; hence ABCDE : *abcde* : : XYZ : *xyz*; hence the sections *abcde*, *xyz* are to each other as the bases ABCDE, XYZ.

Cor. 2. If the bases ABCDE, XYZ are equivalent, any sections *abcde*, *xyz*, made at equal distances from those bases, will be equivalent likewise.

PROPOSITION XVII. THEOREM.

Two triangular pyramids, having equivalent bases and equal altitudes, are equivalent, or equal in solidity.





Let SABC, *sabc* be those two pyramids, of which the two bases ABC, *abc*, conceived to be situated in the same plane, are equivalent, the altitude TA being the same in both. If they are not equivalent, let *sabc* be the smaller; and suppose A*a* to be the altitude of a prism, which having ABC for its base, is equal to their difference.

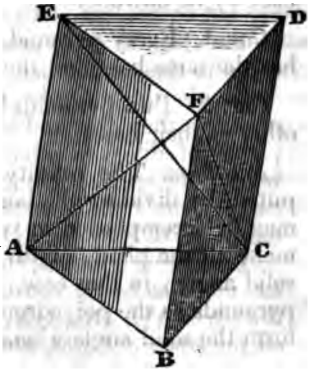
Divide the altitude AT into equal parts A*x*, *xy*, *yz*, &c. each less than A*a*, and let *k* be one of those parts; through the points of division extend planes parallel to the plane of the bases; the corresponding sections formed by these planes in the two pyramids will be respectively equivalent by the last Corollary, namely, DEF to *def*, GHI to *ghi*, &c.

This being granted, upon the triangles ABC, DEF, GHI, &c. taken as bases, construct exterior prisms having for edges the parts AD, DG, GK, &c. of the side SA; in like manner, on the bases *def*, *ghi*, *klm*, &c. in the second pyramid construct interior prisms having for edges the corresponding parts of *sa*. It is plain that the sum of all the exterior prisms of the pyramid SABC will be greater than this pyramid; and also that the sum of all the interior prisms of the little pyramid *sabc* will be less than this. Hence the difference, between the sum of all the exterior prisms and the sum of all the interior ones, must be greater than the difference between the two pyramids themselves.

Now, beginning with the bases ABC, *abc*, the second exterior prism DEFG is equivalent to the first interior prism *defa*, because they have the same altitude *k*, and their bases DEF, *def* are equivalent; for like reasons, the third exterior prism GHIK and the second interior prism *ghid* are equivalent; the fourth exterior and the third interior; and so on, to the last in each series. Hence all the exterior prisms of the pyramid SABC, excepting the first prism DABC, have equivalent corresponding ones in the interior prisms of the pyramid *sabc*: hence the prism DABC is the difference between the sum of all the exterior prisms of the pyramid SABC, and the sum of all the interior prisms of the pyramid *sabc*. But the difference between those two sets of prisms has already been proved to be greater than that of the two pyramids; which latter difference we supposed to be equal to the prism *a*ABC: hence the prism DABC must be greater than the prism *a* ABC. But in reality it is less; for they have the same base ABC, and the altitude A*x* of the first is less than A*a* the altitude of the second. Hence the supposed inequality between the two pyramids cannot exist; hence the two pyramids SABC, *sabc*, having equal altitudes and equivalent bases, are themselves equivalent.

PROPOSITION XVIII. THEOREM.

Every triangular pyramid is a third part of the triangular prism having the same base and the same altitude.



Let FABC be a triangular pyramid, ABCDEF a triangular prism of the same base and the same altitude; the pyramid will be equal to a third of the prism.

Cut off the pyramid FABC from the prism, by a section made along the plane FAC; there will remain the solid FACDE, which may be considered as a quadrangular pyramid, whose vertex is F, and whose base is the parallelogram ACDE. Draw the diagonal CE; and extend the plane FCE, which will cut the quadrangular pyramid into two triangular ones FACE, FCDE. These two triangular pyramids have for their common altitude the perpendicular let fall from F on the plane ACDE; they have equal bases, the triangles ACE, CDE being halves of the same parallelogram; hence (17. VI.) the two pyramids FACE, FCDE are equivalent. But the pyramid FCDE and the pyramid FABC have equal bases ABC, DEF; they have also the same altitude, namely, the distance of the parallel planes ABC, DEF; hence the two pyramids are equivalent. Now the pyramid FCDE has already been proved equivalent to FACE; hence the three pyramids FABC, FCDE, FACE, which compose the prism ABD are all equivalent. Hence the pyramid FABC is the third part of the prism ABD, which has the same base and the same altitude.

Cor. The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

PROPOSITION XIX. THEOREM.

Any pyramid SABCDE *is measured by the third part of the product of its base by its altitude.*

For, extending the planes SEB, SEC (see the figure of Prop. 16.) through the diagonals EB, EC, the polygonal pyramid SABCDE will be divided into several triangular pyramids all having the same altitude SO. But (18. VI. Cor.) each of these pyramids is measured by multiplying its base ABE, BCE, or CDE, by the third part of its altitude SO; hence the sum of these triangular pyramids, or the polygonal pyramid SABCDE will be measured by the sum of the triangles ABE, BCE, CDE, or the polygon ABCDE, multiplied by SO; hence every pyramid is measured by a third part of the product of its base by its altitude.

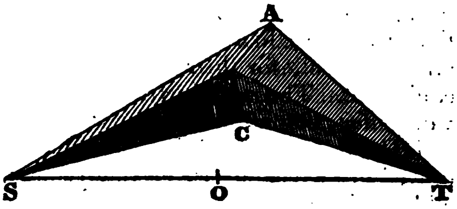
Cor. 1. Every pyramid is the third part of the prism which has the same base and the same altitude.

Cor. 2. Two pyramids having the same altitude are to each other as their bases.

Scholium. The solidity of any polyedral body may be computed, by dividing the body into pyramids; and this division may be accomplished in various ways. One of the simplest is to make all the planes of division pass through the vertex of one solid angle; in that case, there will be formed as many partial pyramids as the polyedron has faces, *minus* those faces which form the solid angle whence the planes of division proceed.

PROPOSITION XX. THEOREM.

Two symmetrical polyedrons are equivalent, or equal in solidity.



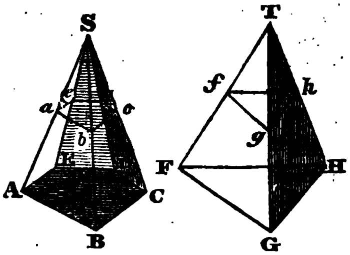
For, in the first place, two symmetrical triangular pyramids, such as SABC, TABC, are both measured by the product of their base ABC and the third part of their altitude SO or TO; hence these pyramids are equivalent.

In the second place, if one of the two symmetrical polyedrons is divided in any way into triangular pyramids, the other may always be divided in the same way into as many symmetrical triangular pyramids; and these symmetrical pyramids are equivalent each to each; hence the whole polyedrons are equivalent, or equal in solidity.

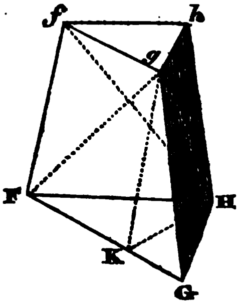
Scholium. This Theorem might seem to result immediately from Prop. 2., where it was shewn that, in two symmetrical polyedrons, all the constituent parts of the one solid are respectively equal to the constituent parts of the other: but it appeared necessary, notwithstanding, to demonstrate this truth in a rigorous manner.

PROPOSITION XXI. THEOREM.

If a pyramid is cut by a plane parallel to its base, the frustum that remains when the little pyramid is taken away, is equal to the sum of three pyramids having for their common altitude the altitude of the frustum, and whose bases are the lower base of the frustum, the upper one, and a mean proportional between the two bases.



Let ABCDE be a pyramid cut by the plane *abd*, parallel to its base; let TFGH be a triangular pyramid whose base and altitude are equal or equivalent to those of the pyramid SABCDE. The two bases may be regarded as situated in the same plane; in which case, the plane *abd*, if produced, will form in the triangular pyramid a section fgh, situated at the same altitude above the common plane of the bases; and therefore (16.VI.) the section fgh will be to the section abd as the base FGH is to the base ABD, and since the bases are equivalent, the sections will be so likewise. Hence the pyramids S$abcde$, Tfgh are equivalent, for their height is the same and their bases are equivalent. The whole pyramids SABCDE, TFGH are equivalent for the same reason; hence the frustums ABDdab, FGH$hfgh$ are equivalent: hence if our proposition can be proved in the single case of the frustum of a triangular pyramid, it will be true of every other.



Let FGH$hfgh$ be the frustum of a triangular pyramid, having parallel bases: through the three points F, g, H, extend the plane FgH; it will cut off from the frustum the triangular pyramid gFGH. This pyramid has for its base the lower base FGH of the frustum; its altitude likewise is that of the frustum, because the vertex g lies in the plane of the upper base fgh.

This pyramid being cut off, there will remain the quadrangular pyramid gfhHF, whose vertex is g, and base fhHF. Extend the plane fgH through the three points f, g, H; it will divide the quadrangular pyramid into two triangular pyramids gFfH, gfhH. The latter has for its base the upper base gfh of the frustum; and for altitude the altitude of the frustum, because its vertex H lies in the lower base. Thus we already know two of the three pyramids which compose the frustum.

It remains to examine the third gFfH. Now, if gK be drawn parallel to fF, and if we conceive a new pyramid fFHK, having K for its vertex and FfH for its base, these two pyramids will have the same base FfH; they will also have the same altitude, because their vertices g and K lie in one line gK, parallel to Ff, and consequently parallel to the plane of the base: hence these pyramids are equivalent. But the pyramid fFKH may be regarded as having its vertex in f, and thus its altitude will be the same as that of the frustum: as to its base FKH, we are now to shew that this is a mean proportional between the bases FGH and fgh. Accordingly, the triangles FHK, fgh have each an equal angle, F $= f$, and an equal side, FK $= fg$; hence (24. III.) we have FHK : fgh : : FH : fh. We have also FHG : FHK : : FG : FK or fg. But the similar triangles FGH, fgh give FG : fg : : FH : fh; hence FGH : FHK : : FHK : fgh; or the base FHK is a mean proportional between the two bases FGH, *fgh*. Hence the frustum of a triangular pyramid, with parallel bases, is equivalent to three pyramids whose common altitude is that of the frustum, and whose bases are the lower base of the frustum, the upper base, and a mean proportional between the two bases.

PROPOSITION XXII. THEOREM.

If a triangular prism, having ABC *for its base, is cut by a plane* DEF *inclined to this base, the solid* ABCDEF, *which results from the section, will be equal to the sum of the three pyramids whose vertices are* D, E, F, *and whose common base is* ABC.

Through the three points F, A, C, (see fig. of Prop. 18.) extend the plane FAC; it will cut off, from the truncated prism ABCDEF, the triangular pyramid FABC: this pyramid has ABC for its base, and the point F for its vertex.

This pyramid being cut off, there will remain the quadrangular pyramid FACDE, whose vertex is F, and whose base is ACDE. Through the three points F, E, C, draw another plane FEC; it will divide the quadrangular pyramid into two triangular pyramids FACE, FCDE.

The pyramid FACE, which has for its base the triangle AEC and for its vertex the point F, is equivalent to a pyramid EABC having AEC for its base and for its vertex the point B. For, these two pyramids have the same base; they have also the same altitude, because the line BF, parallel to each of the lines AE, CD, is parallel to their plane ACE: hence the pyramid FAEC is equivalent to the pyramid EABC; which latter may be considered as having ABC for its base, and the point E for its vertex.

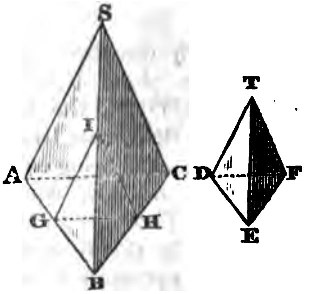
Again, the third pyramid FCDE may in the first place be changed into AFCD; for these two pyramids have the same base FCD; they have also the same altitude, AE being parallel to the plane FCD; hence the pyramid FCDE is equivalent to the pyramid AFCD. In the second place, the pyramid AFCD may be changed into ABCD: for these two pyramids have the common base ACD; they have the same altitude, since their vertices F and B lie in a line parallel to the plane of the base. Hence the pyramid FCDE, equivalent to AFCD, is also equivalent to ABCD; which latter may be regarded as having ABC for its base, and the point D for its vertex.

Hence finally, the truncated prism ABCDEF is equal to the sum of three pyramids whose common base is ABC, and whose vertices are respectively the points D, E, F.

Cor. If the edges AE, BF, CD are perpendicular to the plane of the base, they will at the same time be the altitudes of the three pyramids which compose the truncated prism; so that the solidity of the truncated prism will be expressed by $\frac{1}{3}$ ABC × AE + $\frac{1}{3}$ ABC × BF + $\frac{1}{3}$ ABC × CD, or which is the same thing, by $\frac{1}{3}$ ABC × (AE + BF + CD).

PROPOSITION XXIII. THEOREM.

Two similar triangular pyramids have their homologous faces similar, and their solid angles equal.



By the Definition, the two triangular pyramids SABC, TDEF are similar, when the two triangles SAB, ABC are similar to the two TDE, DEF, and similarly placed, that is to say, when the angle ABS=DET, BAS=EDT, ABC =DEF, BAC=EDF, and when also the inclination of the planes SAB, ABC is equal to that of the planes TDE, DEF. This being granted, we are to shew that these pyramids have all their faces similar each to each, and their homologous solid angles equal.

Take BC=ED, BH=EF, BI=ET; and join GH, GI, IH. The pyramid TDEF is equal to the pyramid IGBH; for, the sides GB, BH being made equal to the sides DE, EF, and the angle GBH being equal to the angle DEF by hypothesis, the triangle GBH must be equal to DEF; hence, to bring about the superposition of the two pyramids, the base DEF may in the first place be laid on its equal GBH; then, because the plane DTE is inclined to DEF as much as the plane SAB to ABC, it is evident that the plane DET will fall indefinitely on the plane ABS. But, by hypothesis, the angle DET=GBI; therefore ET will fall on its equal BI: and since the four points D, E, F, T, coincide with the four G, B, H, I, it follows (1. VI.) that the pyramid TDEF coincides with the pyramid IGBH.

Now, by reason of the equal triangles DEF, GBH, we have the angle BGH=EDF=BAC; hence GH is parallel to AC. For a like reason, GI is parallel to AS; hence (13. V.) the plane

IGH is parallel to SAC. It follows, therefore (15. VI.), that the triangle IGH or its equal TDF is similar to SAC, and the triangle IBH or its equal TEF to SBC; hence the two similar triangular pyramids SABC, TDEF have their four faces similar each to each. We are now to shew that the homologous solid angles are equal.

The solid angle E has already been laid on its homologous one B; and the same thing might be done with any other two homologous solid angles: but it is clear of itself that any two homologous solid angles, T and S for example, are equal, because they are formed by three plane angles, which are respectively equal and similarly placed.

Hence two similar triangular pyramids have their homologous faces similar, and their homologous solid angles equal.

Cor. 1. The similar triangles in their two pyramids furnish the proportions, AB : DE :: BC : EF :: AC : DF :: AS : DT :: SB : TE :: SC : TF; hence, *in similar triangular pyramids, the homologous sides are proportional.*

Cor. 2. And since the homologous solid angles are equal, it follows, that *the inclination of any two faces in a pyramid is equal to the inclination of the corresponding two in the similar pyramid.*

Cor. 3. If the triangular pyramid SABC is cut by a plane GIH parallel to SAC one of its faces, the partial pyramid BGIH will be similar to the whole pyramid BASC: for, the triangles BGI, BGH are similar to the triangles BAS, BAC, each to each, and similarly placed; the inclination of their planes is the same in both; hence the two pyramids are similar.

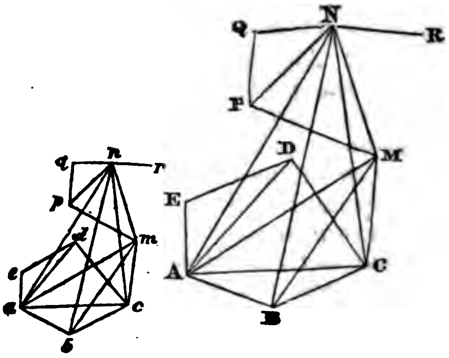
Cor. 4. In general, (see the fig. of Prop. 19. VI.) *if any pyramid* SABCDE *is cut by a plane* abcde *parallel to the base, the partial pyramid* Sabcde *will be similar to the whole pyramid* SABCDE. For the bases ABCDE, *abcde* are similar; and, joining AC, *ac*, we have just proved that the triangular pyramid SABC must be similar to the pyramid S*abc*; hence (*Def.* 18. VI.) the point S in reference to the base ABC, is determined exactly as the point S in reference to the base *abc*; hence the two pyramids SABCDE, S*abcde* are similar.

Scholium. Instead of the five data required by the Definition to determine the similarity of two triangular pyramids, five other data might be substituted according to various combinations, and there would result as many theorems as combinations. Among these, the following deserves to be specially noticed: *Two triangular pyramids are similar when they have their homologous sides proportional.*

For (see the figure of our Proposition), if we have the proportions AB : DE :: BC : EF :: AC : DF :: AS : DT :: SB : TE :: SC : TF, which includes five conditions, the triangles ABS, ABC will be respectively similar to the triangles DET, DEF, and similarly placed. The triangle SBC will be similar to TEF; hence the three plane angles which form the solid angle B will be equal to the plane angles which form the solid angle E, each to each; hence the inclination of the planes SAB, ABC is equal to that of their homologous planes TDE, DEF; hence the two pyramids are similar.

PROPOSITION XXIV. THEOREM.

Two similar polyedrons have their homologous faces similar, and their homologous solid angles equal.



Let ABCDE be the base of a polyedron, let M and N be the vertices of two solid angles, without this base, determined by the triangular pyramids MABC, NABC, whose common base is ABC; let *abcde* in the other polyedron be the base homologous or similar to ABCDE; *m* and *n* the vertices homologous to M and N, and determined by the pyramids *mabc*, *nabc*, similar to the pyramids MABC, NABC: we are to shew, first, that the distances MN, *mn*, are proportional to the homologous sides AB, *ab*.

The pyramids MABC, *mabc* being similar, the inclination of the planes MAC, BAC is equal to that of the planes *mac*, *bac*; in like manner, the pyramids NABC, *nabc* being similar, the inclination of the planes NAC, BAC is equal to that of the planes *nac*, *bac*: and taking away the two former inclinations from the two latter, there will remain the inclination of the planes NAC, MAC equal to that of the planes *nac*, *mac*. But farther, since these same pyramids are similar, the triangle MAC is similar to *mac*; and the triangle NAC to *nac*: hence the two triangular pyramids MNAC, *mnac* have two faces in each respectively similar, similarly placed, and equally inclined;

hence (21. VI.) these two pyramids are similar; and their homologous sides give the proportion MN : *mn* :: AM : *am*. Besides, AM : *am* :: AB : *ab*; hence MN : *mn* : : AB : *ab*.

Let P and *p* be two other homologous vertices of the same polyedrons, we shall in like manner have PN : *pn* : : AB : *ab*, and PM : *pm* : : AB : *ab*. Hence MN : *mn* :: PN : *pn* :: PM : *pm*. Hence *the triangle* PNM, *which joins any three vertices of the one polyedron, is similar to the triangle* pnm *which joins the three homologous vertices of the other.*

Again, let Q and *q* be two homologous vertices; and the triangle PQN will be similar to *pqn*. We assert farther, that the inclination of the planes PQN, PMN is equal to that of the planes *pqn*, *pmn*.

For, joining QM and *qm*, we shall still have the triangle QNM similar to *qnm*, and therefore the angle QNM equal to *qnm*. Imagine a solid angle to be formed at N, by the three plane angles QNM, QNP, PNM; and another solid angle to be formed at *n*, by the three plane angles *qnm*, *qnp*, *pnm*. Since these plane angles are respectively equal, the solid angles must be equal also. Hence the inclination of the two planes PNQ, PNM is equal to that of their homologous planes *pnq*, *pnm*; hence, if the two triangles PNQ, PNM were in the same plane, in which case, the angle QNM would be equal to QNP+PNM, we should likewise have the angle *qnm* equal to *qnp*+*pnm*, and the two triangles *qnp*, *pnm* would also be in the same plane.

All that we have now proved is true, whatever be the value of the angles M, N, P, Q, compared with their corresponding ones *m*, *n*, *p*, *q*.

Let us next suppose the surface of one of the polyedrons to be divided into triangles ABC, ACD, MNP, NPQ, &c.; the surface of the other polyedrons will evidently contain an equal number of triangles *abc*, *acd*, *mnp*, *npq*, *&c.*, similar and similarly placed; and if several triangles, as MPN, NPQ, &c. belong to one face, and lie in the same plane, their corresponding triangles *mpn*, *npq*, *&c.* will likewise lie in one plane. Hence every polygonal face in the one polyedron will correspond to a similar polygonal face in the other polyedron; hence the two polyedrons will be terminated by the same number of planes similar, and similarly placed. We assert farther, that their homologous solid angles will be equal.

For, if the solid angle N, for example, is formed by the plane angles QNP, PNM, MNR, QNR, the corresponding solid angle *n* will be formed by the plane angles *qnp*, *pnm*, *mnr*, *qnr*. But these plane angles are equal each to each; and the inclination of any two adjacent planes is equal to that of the two which correspond to them: hence the two solid angles are equal, because they would exactly coincide.

Hence, finally, two similar polyedrons have their homologous faces similar and their homologous solid angles equal.

Cor. From the preceding demonstration, it follows, that if a triangular pyramid were formed with four vertices of one polyedron, and a second with the four corresponding vertices of a similar polyedron, these two pyramids (21. VI. Schol.) would be similar, because their homologous sides would be proportional.

It is also evident that two homologous diagonals (17. II.), as AN, *an*, are to each as two homologous sides AB, *ab*.

PROPOSITION XXV. THEOREM.

Two similar polyedrons may be divided into the same number of triangular pyramids, similar each to each, and similarly placed.

For, it has already been shewn, that the surfaces of two polyedrons may be divided into the same number of triangles similar each to each, and similarly placed. Consider all the triangles of the one polyedron, except those which form the solid angle A, as the bases of so many triangular pyramids having their common vertex in A; those pyramids taken together will compose the whole polyedron. Divide the other polyedron, in the same manner, into pyramids having for their common vertex the angle *a*, homologous to A: the pyramid, which joins four vertices of the one polyedron, will evidently be similar to the pyramid which joins the four homologous vertices of the other polyedron. Hence two similar polyedrons, &c.

PROPOSITION XXVI. THEOREM.

Two similar pyramids are to each other as the cubes of their homologous sides.

For two pyramids being similar, (see fig. of Prop. 16.) the smaller may be placed within the greater, so that the solid angle S shall be common to both. In that position, the bases ABCDE, *abcde* will be parallel; because, since (22. VI.) the homologous faces are similar, the angle S*ab* is equal to SAB, and S*bc* to SBC; hence (13. V.) the plane ABC is parallel to the plane *abc*. This granted, let SO be the perpendicular drawn from the vertex S to the plane ABC, and *o* the point where this perpendicular meets the plane *abc*: from what has already been shewn (15. VI.), we shall have SO : S*o* : : SA : S*a* : : AB : *ab*; and consequently,

$$\tfrac{1}{3}SO : \tfrac{1}{3}So :: AB : ab.$$

But the bases ABCDE, *abcde* being similar figures, we have

$$ABCDE : abcde :: AB^2 : ab^2.$$

Multiply the corresponding terms of these two propositions; there results the proportion,

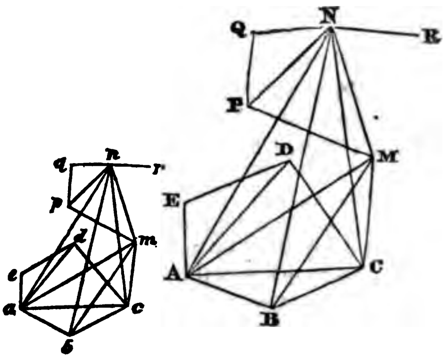
$$ABCDE\times\tfrac{1}{3}SO :: abcde \times\tfrac{1}{3}So :: AB^3 : ab^3.$$

Now $ABCDE\times\frac{1}{3}SO$ is the solidity of the pyramid SABCDE, and $abcde\times\frac{1}{3}So$ is that of the pyramid S*abcde* (18. VI.); hence two similar pyramids are to each other as the cubes of their homologous sides.

PROPOSITION XXVII. THEOREM.

Two similar polyedrons are to each other as the cubes of their homologous sides.

For two similar polyedrons may (23. VI.) be divided, each into the same number of triangular pyramids respectively similar. Now, the two similar pyramids APNM, *apnm* are to each other as the cubes of their homologous sides AM, *am*, or as the cubes of their homologous sides AB, *ab*. The same ratio exists between every other pair of homologous pyramids; hence the sum of all the pyramids which compose the one polyedron, or that polyedron itself, is to the other polyedron, as the cube of any one side in the first is to the cube of the homologous side in the second.



General Scholium.

The chief propositions of this Book relating to the solidity of polyedrons may be exhibited in algebraical terms, and so recapitulated in the briefest manner possible.

Let B represent the base of a *prism*; H its altitude: the solidity of the prism will be B×H, or BH.

Let B represent the base of a *pyramid*; H its altitude: the solidity of the pyramid will be $B\times\frac{1}{3}H$, or $H\times\frac{1}{3}B$, or $\frac{1}{3}BH$.

Let H represent the altitude of the *frustum of a pyramid*, having parallel bases A and B; $\sqrt{AB}$ will be the mean proportional between those bases; and the solidity of the frustum will be $\frac{1}{3}H\times(A+B+\sqrt{AB})$.

Let B represent the base of the *frustum of a triangular prism*; H, H′, H″ the altitudes of its three upper vertices: the solidity of the truncated prism will be $\frac{1}{3}B\times(H+H'+H'')$.

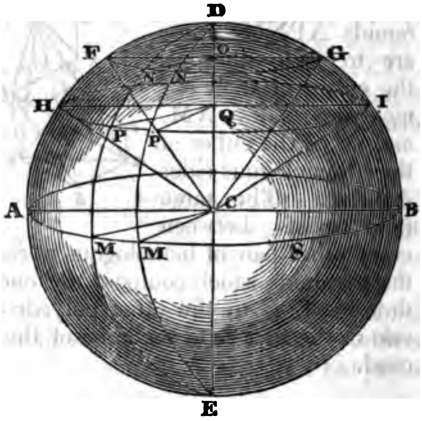
In fine, let P and p represent the *solidities of two similar polyedrons*; A and a two homologous sides or diagonals of these polyedrons: we shall have $P : p :: A^3 : a^3$.

BOOK VII.

THE SPHERE.

Definitions.

I. The *sphere* is a solid terminated by a curve surface, all the points of which are equally distant from a point within, called the *centre*.



The sphere may be conceived to be generated by the revolution of a semicircle DAE about its diameter DE; for the surface described in this movement, by the curve DAE, will have all its points equally distant from the centre C.

II. The *radius of a sphere* is a straight line drawn from the centre to any point in the surface; the *diameter* or *axis* is a line passing through this centre, and terminated on both sides by the surface.

All the radii of a sphere are equal; all the diameters are equal, and double of the radius.

III. It will be shewn (1. VII.) that every section of the sphere, made by a plane, is a circle: this granted, a *great circle* is a section which passes through the centre; a *little circle* one which does not pass through it.

IV. A *plane* is a *tangent* to a sphere, when their surfaces have but one point in common.

V. The *pole of a circle* of a sphere is a point in the surface equally distant from all the points in the circumference of this circle. It will be shewn (6. VII.) that every circle, great or little, has always two poles.

VI. A *spherical triangle* is a portion of the surface of a sphere, bounded by three arcs of great circles.

Those arcs, named the *sides* of the triangle, are always supposed to be each less than a semicircumference. The angles, which their planes form with each other, are the angles of the triangle.

VII. A spherical triangle takes the name of *right-angled, isosceles, equilateral,* in the same cases as a rectileneal triangle.

VIII. A *spherical polygon* is a portion of the surface of a sphere terminated by several arcs of great circles.

IX. A *lune* is that portion of the surface of a sphere, which is included between two great semicircles meeting in a common diameter.

X. A *spherical wedge* or *ungula* is that portion of the solid sphere, which is included between the same great semicircles, and has the lune for its base.

XI. A *spherical pyramid* is a portion of the solid sphere, included between the planes of a solid angle whose vertex is the centre. The *base* of the pyramid is the spherical polygon intercepted by the same planes.

XII. A *zone* is the portion of the surface of the sphere, included between two parallel planes, which form its *bases*. One of those planes may be a tangent to the sphere; in which case, the zone has only a single base.

XIII. A *spherical segment* is the portion of the solid sphere, included between two parallel planes which form its bases.

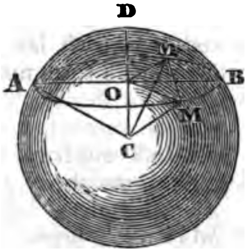
One of those planes may be a tangent to the sphere; in which case, the segment has only a single base.

XIV. The *altitude of a zone* or *of a segment* is the distance of the two parallel planes, which form the bases of the zone or segment.

XV. Whilst the semicircle DAE (see Def. 1.) revolving round its diameter DE, describes the sphere; any circular sector, as DCF or FCH, describes a solid, which is named a *spherical sector*.

PROPOSITION I. THEOREM.

Every section of a sphere, made by a plane, is a circle.



Let AMB be the section, made by a plane, in the sphere whose centre is C. From the point C, draw CO perpendicular to the plane AMB; and different lines CM, CM to different points of the curve AMB, which terminates the section.

The oblique lines CM, CM, CB are equal, being radii of the sphere; hence (5. V.) they are equally distant from the perpendicular CO; hence all the lines OM, MO, OB are equal; hence the section AMB is a circle, whose centre is O.

Cor. 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence all great circles are equal.

Cor. 2. Two great circles always bisect each other; for their common intersection, passing through the centre, is a diameter.

Cor. 3. Every great circle divides the sphere and its surface into two equal parts: for, if the two hemispheres were separated, and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide, no point of the one being nearer the centre than any point of the other.

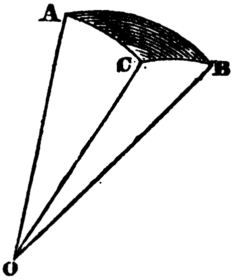
Cor. 4. The centre of a little circle, and that of the sphere, are in the same straight line, perpendicular to the plane of the little circle.

Cor. 5. Little circles are the less the farther they lie from the centre of the sphere; for the greater CO is, the less is the chord AB, the diameter of the little circle AMB.

Cor. 6. An arc of a great circle may always be made to pass through any two given points in the surface of the sphere; for the two given points and the centre of the sphere make three points, which determine the position of a plane. But if the two given points were at the extremities of a diameter, these two points and the centre would then lie in one straight line, and an infinite number of great circles might be made to pass through the two given points.

PROPOSITION II. THEOREM.

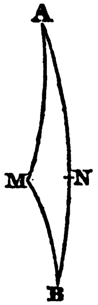
In every spherical triangle ABC, *any side is less than the sum of the other two.*



Let O be the centre of the sphere; and draw the radii OA, OB, OC. Imagine the planes AOB, AOC, COB; those planes will form a solid angle at the point O; and the angles AOB, AOC, COB will be measured by AB, AC, BC, the sides of the spherical triangle. But (21. V.) each of the three plane angles composing a solid angle is less than the sum of the other two; hence any side of the triangle ABC is less than the sum of the other two.

PROPOSITION III. THEOREM.

The shortest path from one point to another, on the surface of a sphere, is the arc of the great circle which joins the two given points.



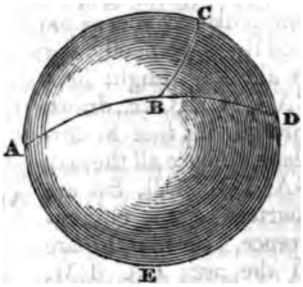
Let ANB be the arc of the great circle which joins the points A and B; and without this line, if possible, let M be a point of the shortest path between A and B. Through the point M, draw MA, MB, arcs of great circles; and take BN=MB.

By the last theorem, the arc ANB is shorter than AM + MB; take BN=BM respectively from both; there will remain AN<AM. Now, the distance of B from M, whether it be the same with the arc BM or with any other line, is equal to the distance of B from N; for by making the plane of the great circle BM to revolve about the diameter which passes through B, the point M may be brought into the position of the point N; and the shortest line between M and B, whatever it may

be, will then be identical with that between N and B: hence the two paths from A to B, one passing through M, the other through N, have an equal part in each, the part from M to B equal to the part from N to B. The first path is the shorter, by hypothesis; hence the distance from A to M must be shorter than the distance from A to N; which is absurd, the arc AM being proved greater than AN: hence no point of the shortest line from A to B can lie out of the arc ANB; hence this arc is itself the shortest distance between its two extremities.

PROPOSITION IV. THEOREM.

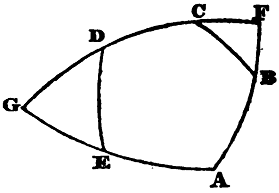
The sum of all the three sides of a spherical triangle is less than the circumference of a great circle.



Let ABC be any spherical triangle; produce the sides AB, AC till they meet again in D. The arcs ABD, ACD will be semicircumferences, since (1. VII. Cor.) two great circles always bisect each other. But in the triangle BCD, we have (2. VII.) the side $BC < BD+CD$; add $AB+AC$ to both; we shall have $AB+AC+BC < ABD+ACD$, that is to say, less than a circumference.

PROPOSITION V. THEOREM.

The sum of all the sides of any spherical polygon is less than the circumference of a great circle.

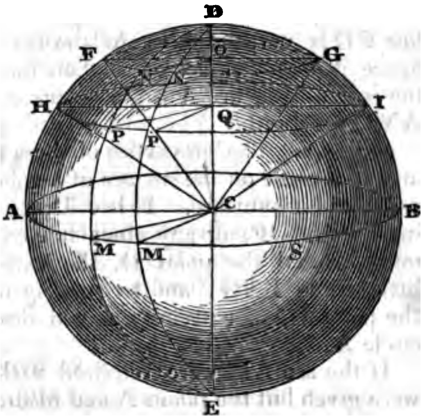


Take the pentagon ABCDE, for example. Produce the sides AB, DC, till they meet in F; then since BC is less than $BF+CF$, the perimeter of the pentagon ABCDE will be less than that of the quadrilateral AEDF. Again, produce the sides AE, FD, till they meet in G; we shall have $ED < EG+DG$; hence the perimeter of the quadrilateral AEDF is less than that of the triangle AFG; which last is itself less than the circumference of a great circle; hence *a fortiori* the perimeter of the polygon ABCDE is less than this same circumference.

Scholium. This proposition is fundamentally the same as Prop. 22. V.; for, O being the centre of the sphere, a solid angle may be conceived as formed at O by the plane angles AOB, BOC, COD, &c., and the sum of these angles must be less than four right angles; which is exactly the proposition we have been engaged with. The demonstration here given is different from that of Book V.; both, however, suppose that the polygon ABCDE is convex, or that no side produced will cut the figure.

PROPOSITION VI. THEOREM.

The diameter DE *being drawn perpendicular to the plane of the great circle* AMB, *the extremities* D *and* E *of this diameter will be the poles of the circle* AMB, *and of all the little circles, as* FNG, *which are parallel to it.*



For, DC being perpendicular to the plane AMB, is perpendicular to all the straight lines CA, CM, CB, &c., drawn through its foot in this plane; hence all the arcs DA, DM, DB, &c. are quarters of the circumference. So likewise are all the arcs EA, EM, EB, &c.; hence the points D and E are each equally distant from all the points of the circumference AMB; hence (*Def.* 5. VII.) they are the poles of that circumference.

Again, the radius DC, perpendicular to the plane AMB, is perpendicular to its parallel FNG; hence (1. VII.) it passes through O the centre of the circle FNG; hence, if the oblique lines DF, DN, DG be drawn, these oblique lines will diverge equally from the perpendicular DO, and will themselves be equal. But, the chords being equal, the arcs are equal; hence the point D is the pole of the little circle FNG; and for like reasons, the point E is the other pole.

Kor. 1. Every arc DM, drawn from a point in the arc of a great circle AMB to its pole, is a quarter of the circumference, which, for the sake of brevity, is usually named a *quadrans* or quadrant; and this quadrant at the same time makes a right angle with the arc AM. For (18. VI.) the line DC being perpendicular to the plane AMC, every plane DMC passing through the line DC is perpendicular to the plane AMC; hence the angle of these planes, or (*Def.* 6. VII.) the angle AMD, is a right angle.

Cor. 2. To find the pole of a given arc AM, draw the indefinite arc MD perpendicular to AM; take MD equal to a quadrant; the point D will be one of the poles of the arc AMD: or thus, at the two points A and M, draw the arcs AD and MD perpendicular to AM; their point of intersection D will be the pole required.

Cor. 3. Conversely, if the distance of the point D from each of the points A and M is equal to a quadrant, the point D will be the pole of the arc AM, and also the angles DAM, AMD will be right.

For, let C be the centre of the sphere; and draw the radii CA, CD, CM. Since the angles ACD, MCD are right, the line CD is perpendicular to the two straight lines CA, CM; hence it is perpendicular to their plane; hence the point D is the pole of the arc AM; and consequently the angles DAM, AMD are right.

Scholium. The properties of these poles enable us to describe arcs of a circle on the surface of a sphere, with the same facility as on a plane surface. It is evident, for instance, that by turning the arc DF, or any other line extending to the same distance, round the point D, the extremity F will describe the little circle FNG; and by turning the quadrant DFA round the point D, its extremity A will describe the arc of the great circle AM.

If the arc AM were required to be produced, and nothing were given but the points A and M through which it was to pass, we should first have to determine the pole D, by the intersection of two arcs described from the points A and M as centres, with a distance equal to a quadrant; the pole D being found, we might describe the arc AM and its prolongation, from D as a centre, and with the same distance as before.

In fine, if it is required from a given point P to let fall a perpendicular on the given arc AM; produce this arc to S, till the distance PS be equal to a quadrant; then from the pole S, and with the same distance, describe the arc PM, which will be the perpendicular required.

PROPOSITION VII. THEOREM.

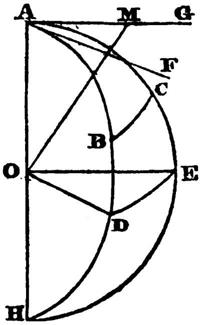
Every plane perpendicular to a radius at its extremity is a tangent to the sphere.

Let FAG (see the next figure) be a plane perpendicular to the radius OA. Any point M in this plane being assumed, and OM, AM being joined, the angle OAM will be right, and hence the distance OM will be greater than OA. Hence the point M lies without the sphere; and as the case is similar with every other point in the plane FAG, this plane can have no point but A common to it with the surface of the sphere; hence (*Def.* 4. VII.) it is a tangent.

Scholium. In the same way it may be shewn, that two spheres have but one point in common, and therefore touch each other, when the distance between their centres is equal to the sum or the difference of their radii; in which case, the centres and the point of contact lie in the same straight line.

PROPOSITION VIII. THEOREM.

The angle BAC, *formed by* AB, AC *two arcs of great circles, is equal to the angle* FAG *formed by the tangents of these arcs at the point* A; *and is therefore measured by the arc* DE *described from the point* A *as a pole, between the sides* AB, AC, *produced if necessary.*



For the tangent AF, drawn in the plane of the arc AB, is perpendicular to the radius AO; and the tangent AG, drawn in the plane of the arc AC, is perpendicular to the same radius AO. Hence (17. V.) the angle FAG is equal to the angle contained by the planes OAB, OAC; which is that of the arcs AB, AC, and is named BAC.

In like manner, if the arcs AD and AE are both quadrants, the lines OD, OE will be perpendicular to AO, and the angle DOE will still be equal to the angle of the planes AOD, AOE; hence the arc DE is the measure of the angle contained by these planes, or of the angle CAB.

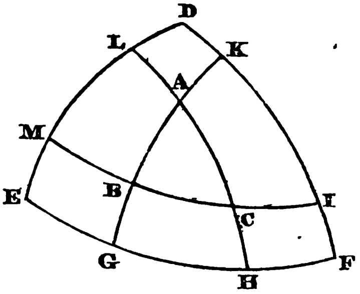
Cor. The angles of spherical triangles may be compared together, by means of the arcs of great circles described from their vertices as poles and included between their sides: hence it is easy to make an angle of this kind equal to a given angle.

Scholium. Vertical angles, such as ACO and BCN (see the figure of Prop. 22. VII.) are equal; for either of them is still the angle formed by the two planes ACB, OCN.

It is farther evident, that, in the intersection of two arcs ACB, OCN, the two adjacent angles ACO, OCB taken together are equal to two right angles.

PROPOSITION IX. THEOREM.

The triangle ABC *being given, if from the points* A, B, C *as poles, the arcs* EF, FD, DE *be described to form the triangle* DEF; *then, conversely, the three points* D, E, F *will be the poles of the sides* BC, AC, AB.



For, the point A being the pole of the arc EF, the distance AE is a quadrant; the point C being the pole of the arc DE, the distance CE is likewise a quadrant: hence the point E is removed the length of a quadrant from each of the points A and C; hence (6. VII. Cor. 3.) it is the pole of the arc AC. It might be shewn, by the same method, that D is the pole of the arc BC, and F that of the arc AB.

Cor. Hence the triangle ABC may be described by means of DEF, as DEF may by means of ABC.

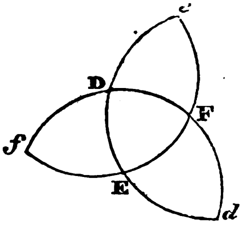
PROPOSITION X. THEOREM.

The same supposition continuing as in the last Theorem, each angle in the one of the triangles, ABC, DEF *will be measured by the semicircumference* minus *the side lying opposite to it in the other triangle.*

Produce the sides (see the preceding figure) AB, AC, if necessary, till they meet EF in G and H. The point A being the pole of the arc GH, the angle A will be measured by that arc. But the arc EH is a quadrant, and likewise GF, E being the pole of AH, and F of AG; hence EH+GF is equal to a semicircumference. Now, EH+GF is the same as EF+GH; hence the arc GH, which measures the angle A, is equal to a semicircumference *minus* the side EF. In like manner, the angle B will be measured by ½ *circ.* — DF: the angle C by ½ *circ.* — DE.

And this property must be reciprocal in the two triangles, since each of them is described in a similar manner by means of the other. Thus we shall find the angles D, E, F of the triangle DEF to be measured respectively by ½ *circ.* — BC, ½ *circ.* — AC, ½ *circ.* — AB. Accordingly the angle D, for example, is measured by the arc MI; but $MI+BC=MC+BI=\frac{1}{2}$ *circ.*; hence the arc MI, the measure of D, is equal to $\frac{1}{2}$ *circ.*$-BC$; and so of all the rest.

Scholium. It must farther be observed, that besides the triangle DEF, three others might be formed by the intersection of the three arcs DE, EF, DF. But the proposition immediately before us is applicable only to the central triangle, which is distinguished from the other three by the circumstance (see the figure in the last page) that the two angles A and D lie on the same side of BC, the two B and E on the same side of AC, and the two C and F on the same side of AB.

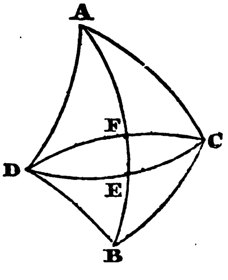


Various names have been given to the triangles ABC, DEF; we shall call them *polar triangles.*

PROPOSITION XI. LEMMA.

The triangle ABC *being given, if from the pole* A, *with a distance* AC, *the arc* DEC *of a little circle be described; if from the pole* B, *with a distance* BC, *the arc* DFC *be described in like manner; and if from the point* D, *where the arcs* DEC, DFC *intersect each other*, AD, DB *two arcs of great circles be drawn; then will* ADB, *the triangle thus formed, have all its parts equal to those of the triangle* ACB.

For, by construction, the side AD=AC, DB=BC, and AB is common; hence those two triangles have their sides equal, each to each. We are now to shew that the angles opposite these equal sides are also equal.



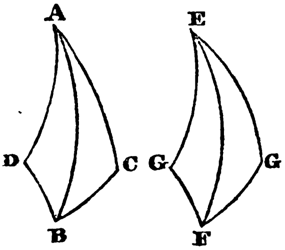
If the centre of the sphere is supposed to be at O, a solid angle may be conceived as formed at O by the three plane angles AOB, AOC, BOC; likewise another solid angle may be conceived as formed by the three plane angles AOB, AOD, BOD. And because the sides of the triangle ABC are equal to those of the triangle ADB, the plane angles forming the one of these solid angles must be equal to the plane angles forming the other, each to each. But in this case we have shewn (23. V.) that the planes, in which the equal angles lie, are equally inclined to each other; hence all the angles of the spherical triangle DAB are respectively equal to those of the triangle CAB, namely, DAB=

BAC, DBA=ABC, and ADB=ACB; hence the sides and the angles of the triangle ADB are equal to the sides and the angles of the triangle ACB.

Scholium. The equality of those triangles is not, however, an absolute equality, or one of superposition; for it would be impossible to apply them to each other exactly, unless they were isosceles. The equality meant here is what we have already named an equality by *symmetry;* therefore we shall call the triangles ACB, ADB, *symmetrical triangles.*

PROPOSITION XII. THEOREM.

Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when they have each an equal angle included between equal sides.



Suppose the side AB=EF, the side AC=EG, and the angle BAC=FEG; the triangle EFG may be placed on the triangle ABC, or on ABD symmetrical with ABC, just as two rectilineal triangles are placed upon each other, when they have an equal angle included between equal sides. Hence all the parts of the triangle EFG will be equal to all the parts of the triangle ABC; that is, besides the three parts equal by hypothesis, we shall have the side BC=FG, the angle ABC=EFG, and the angle ACB=EGF.

PROPOSITION XIII. THEOREM.

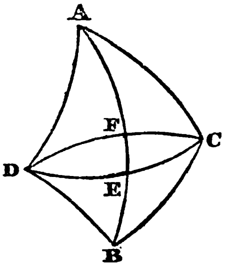
Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when two angles and the interjacent side of the one are equal to two angles and the interjacent side of the other.

For, one of those triangles, or the triangle symmetrical with it, may be placed on the other, as is done in the corresponding case of rectilineal triangles. *See Prop.* 7. I.

PROPOSITION XIV. THEOREM.

If two triangles on the same sphere, or on equal spheres, have all their sides respectively equal, their angles will likewise be all respectively equal, the equal angles lying opposite the equal sides.

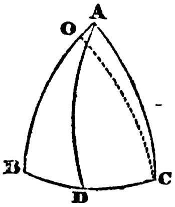
This truth is evident by Prop. 11. VII., where it was shewn that, with three given sides AB, AC, BC, there can only be two triangles ACB, ABD, different as to the position of their parts, and equal as to the magnitude of those parts. Hence those two triangles, having all their sides respectively equal in both, must either be absolutely equal, or at least *symmetrically* so; in both of which cases, their corresponding angles must be equal, and lie opposite to equal sides.



PROPOSITION XV. THEOREM.

In every isosceles spherical triangle, the angles opposite the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle will be isosceles.

First. Suppose the side AB=AC; we shall have the angle C=B. For, if the arc AD be drawn from the vertex A to the middle point D of the base, the two triangles ABD, ACD will have all the sides of the one respectively equal to the corresponding sides of the other, namely, AD common, BD=DC, and AB=AC: hence by the last Proposition, their angles will be equal; therefore B=C.

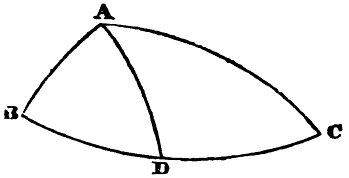


Secondly. Suppose the angle B=C; we shall have the side AC=AB. For, if not, let AB be the greater of the two; take BO=AC, and join OC. The two sides BO, BC are equal to the two AC, BC; the angle OBC, contained by the first two, is equal to ABC contained by the second two. Hence (12. VII.) the two triangles BOC, ACB have all their other parts equal; hence the angle OCB=ABC: but by hypothesis, the angle ABC=ACB; hence we have OCB=ACB, which is absurd; hence it is absurd to suppose AB different from AC; hence the sides AB, AC, opposite to the equal angles B and C, are equal.

Scholium. The same demonstration proves the angle BAD=DAC, and the angle BDA=ADC. Hence the two last are right angles; hence *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is at right angles to that base, and bisects the opposite angle.*

PROPOSITION XVI. THEOREM.

In a spherical triangle ABC, *if the angle* A *is greater than the angle* B, *the side* BC *opposite to* A *will be greater than the side* AC *opposite to* B; *and conversely, if the side* BC *is greater than* AC, *the angle* A *will be greater than the angle* B.

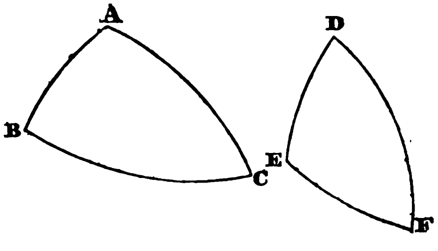


First. Suppose the angle $A>B$; make the angle $BAD=B$: then (15. VII.) we shall have $AD=DB$: but $AD+DC$ is greater than AC; hence, putting DB in place of AD, we shall have $DB+DC$ or $BC>AC$.

Secondly. If we suppose $BC>AC$, the angle BAC will be greater than ABC. For, if BAC were equal to ABC, we should have $BC=AC$; if BAC were less than ABC, we should then, as has just been shewn, find $BC<AC$. Both these conclusions are false: hence the angle BAC is greater than ABC.

PROPOSITION XVII. THEOREM.

If the two sides AB, AC, *of the spherical triangle* ABC, *are equal to the two sides* DE, DF *of the triangle* DEF, *drawn upon an equal sphere; and if at the same time the angle* A *is greater than the angle* D, *then will the third side* BC *of the first triangle be greater than the third side* EF *of the second.*



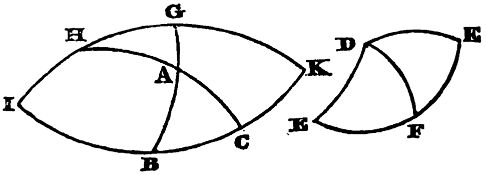
The demonstration is every way similar to that of Prop. 2. I.

PROPOSITION XVIII. THEOREM.

If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they will also be mutually equilateral.

Let A and B be the two given triangles; P and Q their polar triangles. Since the angles are equal in the triangles A and B, the sides will be equal in the polar triangles P and Q (10. VII.): but since the triangles P and Q are mutually equilateral, they must also (14. VII.) be mutually equiangular; and, lastly, the angles being equal in the triangles P and Q, it follows (10. VII.) that the sides are equal in their polar triangles A and B. Hence the mutually equiangular triangles A and B are at the same time mutually equilateral.

This proposition may also be demonstrated, without the aid of polar triangles, as follows:



Let ABC, DEF be two triangles mutually equiangular, having A=D, B=E, C=F; we are to shew that AB=DE, AC=DF, AC=DF, BC=EF.

On the prolongations of the sides AB, AC, take AG=DE, and AH=DF; join GH; and produce the arcs BC, GH, till they meet in I and K.

The two sides AG, AH are equal, by construction, to the two DF, DE; the included angle GAH = BAC = EDF; hence (12. VII.) the triangles AGH, DEF, are equal in all their parts; hence the angle AGH=DEF=ABC, and the angle AHG=DFE=ACB.

In the triangles IBG, KBG, the side BG is common; the angle IBG=GBK; and, since IGB+BGK is equal to two right angles, and likewise GBK + IBG, it follows that BGK=IBG. Hence (13. VII.) the triangles IBG, GBK are equal; hence IG=BK, and IB=GK.

In like manner, the angle AHG being equal to ACB, we can shew that the triangles ICH, HCK have two angles and the interjacent side in each equal; hence they are themselves equal; hence IH=CK, and HK=IC.

Now, if the equals CK, IH be taken away from the equals BK, IG, the remainders BC, GH will be equal. Besides, the angle BCA=AHG, and the angle ABC=AGH. Hence the triangles ABC, AHG have two angles and the interjacent side in each equal; hence they are themselves equal. But the triangle DEF is equal in all its parts to AHG; hence it is also equal to the triangle ABC, and we have AB=DE, AC=DF, BC=EF; hence if two spherical triangles are mutually equiangular, the sides opposite their equal angles will also be equal.

Scholium. This Proposition is not applicable to rectilineal triangles; in which, equality among the angles indicates only proportionality among the sides. Nor is it difficult to account for the difference observable, in this respect, between spherical and rectilineal triangles. In the Proposition now before us, as well as in Propp. 12, 13, 14, 17, which treat of the comparison of triangles, it is expressly required that the arcs be traced on the same sphere, or on equal spheres. Now similar arcs are to each other as their radii; hence, on equal spheres, two triangles cannot be similar without being equal. Therefore it is not strange that equality among the angles should produce equality among the sides.

The case would be different, if the triangles were drawn upon unequal spheres; there, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of their spheres.

PROPOSITION XIX. THEOREM.

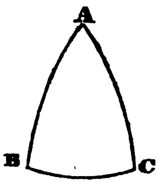
The sum of all the angles in any spherical triangle is less than six right angles, and greater than two.

For, in the first place, every angle of a spherical triangle is less than two right angles (see the following Scholium); hence the sum of all the three is less than six right angles.

Secondly, the measure of each angle in a spherical triangle (10. VII.) is equal to the semicircumference *minus* the corresponding side of the polar triangle; hence the sum of all the three is measured by three semicircumferences *minus* the sum of all the sides of the polar triangle. Now (4. VII.), this latter sum is less than a circumference; therefore, taking it away from three semicircumferences, the remainder will be greater than one semicircumference, which is the measure of two right angles; hence, in the second place, the sum of all the angles in a spherical triangle is greater than two right angles.

Cor. 1. The sum of all the angles in a spherical triangle is not constant, like that of all the angles in a rectilineal triangle; it varies between two right angles and six, without ever arriving at either of these limits. Two given angles therefore do not serve to determine the third.

Cor. 2. A spherical triangle may have two or even three angles right, two or three obtuse.



If the triangle ABC is *bi-rectangular*, in other words, has two right angles B and C, the vertex A will (6. VII.) be the pole of the base BC; and the sides AB, AC will be quadrants.

If the angle A is also right, the triangle ABC will be *tri-rectangular*; its angles will all be right, and its sides quadrants. The tri-rectangular triangle is contained eight times in the surface of the sphere; as is evident from the figure of Prop. 20, supposing the arc MN to be a quadrant.

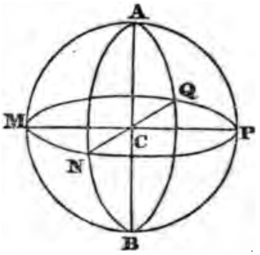
Scholium. In all the preceding observations, we have supposed, in conformity with Definition 6, that our spherical triangles have always each of their sides less than a semicircumference; from which it follows that any one of their angles is always less than two right angles. For (see fig. to Prop. 4.) if the side AB is less than a semicircumference, and AC is so likewise, both those arcs will require to be produced before they can meet in D. Now the two angles ABC, CBD, taken together, are equal to two right angles; hence the angle ABC itself is less than two right angles.

We may observe, however, that some spherical triangles do exist, in which certain of the sides are greater than a semicircumference, and certain of the angles greater than two right angles. Thus, if the side AC is produced so as to form a whole circumference ACE, the part which remains, after subtracting the triangle ABC from the hemisphere, is a new triangle also designated by ABC, and having AB, BC, AEDC for its sides. Here, it is plain, the side AEDC is greater than the semicircumference AED; and, at the same time, the angle B opposite to it exceeds two right angles, by the quantity CBD.

The triangles whose sides and angles are so large have been excluded from our Definition; but the only reason was, that the solution of them, or the determination of their parts, is always reducible to the solution of such triangles as are comprehended by the Definition. Indeed, it is evident enough, that if the sides and angles of the triangle ABC are known, it will be easy to discover the angles and sides of the triangle which bears the same name, and is the difference between a hemisphere and the former triangle.

PROPOSITION XX. THEOREM.

The lune AMBNA *is to the surface of the sphere, as* MAN, *the angle of this lune, is to four right angles, or as the arc* MN, *which measures that angle, is to the circumference.*



Suppose, in the first place, the arc MN to be to the circumference MNPQ as some one rational number is to another, as 5 to 48, for example. The circumference MNPQ being divided into 48 equal parts, MN will contain 5 of them; and if the pole A were joined with the several points of division, by as many quadrants, we should in the hemisphere AMNPQ have 48 triangles, all equal, because having all their parts equal. Hence the whole sphere must contain 96 of those partial triangles, the lune AMBNA will contain 10 of them; hence the lune is to the sphere as 10 is to 96, or as 5 to 48, in other words, as the arc MN is to the circumference.

If the arc MN is not commensurable with the circumference, we may still shew, by a mode of reasoning frequently exemplified already, that in this case also, the lune is to the sphere as MN is to the circumference.

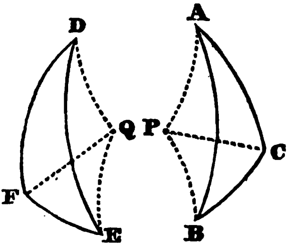
Cor. 1. Two lunes are to each other as their respective angles.

Cor. 2. It was shewn above (19. VII.) that the whole surface of the sphere is equal to eight tri-rectangular triangles; hence, if the area of one such triangles is taken for unity, the surface of the sphere will be represented by 8. This granted, the surface of the lune, whose angle is A, will be expressed by 2A (the angle A being always estimated from the right angle assumed as unity), since 2A : 8 : : A : 4. Thus we have here two different unities; one for angles, being the right angle; the other for surfaces, being the tri-rectangular spherical triangle, or the triangle whose angles are all right, and whose sides are quadrants.

Scholium. The spherical ungula, bounded by the planes AMB, ANB, is to the whole solid sphere, as the angle A is to four right angles. For, the lunes being equal, the spherical ungulas will also be equal; hence two spherical ungulas are to each other, as the angles formed by the planes which bound them.

PROPOSITION XXI. THEOREM.

Two symmetrical spherical triangles are equal in surface.



Let ABC, DEF be two symmetrical triangles, that is to say, two triangles having their sides AB=DE, AC=DF, CB=EF, and yet incapable of coinciding with each other: we are to shew that the surface ABC is equal to the surface DEF.

Let P be the pole of the little circle passing through the three points A, B, C;* from this point draw (6. VII.) the equal arcs PA, PB, PC; at the point F, make the angle DFQ=ACP, the arc FQ=CP; and join DQ, EQ.

The sides DF, FQ are equal to the sides AC, CP; the angle DFQ=ACP: hence (12. VII.) the two triangles DFQ, ACP are equal in all their parts; hence the side DQ=AP, and the angle DQF=APC.

In the proposed triangles DFE, ABC, the angles DFE, ACB, opposite to the equal sides DE, AB, being equal (11. VII.), if the angles DFQ, ACP which are equal by construction, be taken away from them, there will remain the angle QFE, equal to PCB. Also the sides QF, FE are equal to the sides PC, CB; hence the two triangles FQE, CPB are equal in all their parts; hence the side QE=PB, and the angle FQE=CPB.

Now, observing that the triangles DFQ, ACP, which have their sides respectively equal, are at the same time isosceles, we shall see them to be capable of mutual adaptation, when applied to each other; for, having placed PA on its equal QF, the side PC will fall on its equal QD, and thus the two triangles will exactly coincide: hence they are equal; hence the surface DQF=APC. For a like reason, the surface FQE=CPB, and the surface DQE=APB; hence we have DQF+FQE—DQE=APC+CPB—APB, or DFE=ABC; hence the two symmetrical triangles ABC, DEF are equal in surface.

Scholium. The poles P and Q might lie within the triangles ABC, DEF: in which case it would be requisite to add the three triangles DQF, FQE, DQE together, in order to make up the triangle DEF; and in like manner, to add the three triangles APC, CPB, APB together in order to make up the triangle ABC: in all other respects, the demonstration and the result would still be the same.

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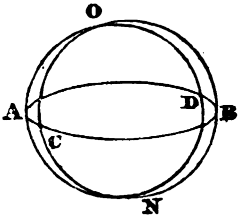
* The circle which passes through the three points A, B, C, or which circumscribes the triangle ABC, can only be a little circle of the sphere; for if it were a great circle, the three sides AB, BC, AC would lie in one plane, and the triangle ABC would be reduced to one of its sides.

PROPOSITION XXII. THEOREM.

If two great circles AOB, COD *intersect each other anyhow in the hemisphere* AOCBD, *the sum of the opposite triangles* AOC, BOD *will be equal to the lune whose angle is* BOD.

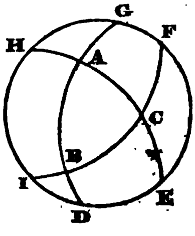


For, producing the arcs OB, OD in the other hemisphere, till they meet in N, the arc OBN will be a semicircumference, and AOB one also; and taking OB from both, we shall have BN=AO. For a like reason, we have DN=CO, and BD=AC. Hence the two triangles AOC, BDN have their three sides respectively equal; besides, they are so placed as to be symmetrical; hence (21. VII.) they are equal in surface, and the sum of the triangles AOC, BOD is equivalent to the lune OBNDO whose angle is BOD.

Scholium. It is likewise evident that the two spherical pyramids, which have the triangles AOC, BOD for bases, are together equivalent to the spherical ungula whose angle is BOD.

PROPOSITION XXIII. THEOREM.

The surface of any spherical triangle is measured by the excess of the sum of its three angles above two right angles.



Let ABC be the proposed triangle: produce its sides till they meet the great circle DEFG drawn anywhere without the triangle. By the last Theorem, the two triangles ADE, AGH are together equivalent to the lune whose angle is A, and which is measured (20. VII.) by 2A. Hence we have ADE+AGH=2A; and for a like reason, BGF+BID=2B, and CIH+CFE=2C. But the sum of those six triangles exceeds the hemisphere by twice the triangle ABC, and the hemisphere is represented by 4; therefore twice the triangle ABC is equal to 2A+2B+2C—4; and consequently once ABC=A+B+C—2; hence every spherical triangle is measured by the sum of all its angles *minus* two right angles.

Cor. 1. However many right angles there be contained in this measure, just so many tri-rectangular triangles, or eighths of the sphere, which (20. VII.) are the unit of surface, will the proposed triangle contain. If the angles, for example, are each equal to $\frac{4}{3}$ of a right angle, the three angles will amount to 4 right angles, and the proposed triangle will be represented by 4—2 or 2; therefore it will be equal to two tri-rectangular triangles, or to the fourth part of the whole sphere's surface.

Cor. 2. The spherical triangle ABC is equivalent to the lune whose angle is $\frac{A+B+C}{2}-1$; likewise the spherical pyramid, which has ABC for its base, is equivalent to the spherical ungula whose angle is $\frac{A+B+C}{2}-1$.

Scholium. While the spherical triangle ABC is compared with the tri-rectangular triangle, the spherical pyramid, which has ABC for its base, is compared with the tri-rectangular pyramid, and a similar proportion is found to subsist between them. The solid angle at the vertex of the pyramid is, in like manner, compared with the solid angle at the vertex of the tri-rectangular pyramid. These comparisons are founded on the coincidence of the corresponding parts. If the bases of the pyramids coincide, the pyramids themselves will evidently coincide, and likewise the solid angles at their vertices. From this, some consequences are deduced.

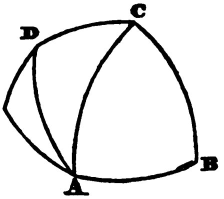
First. Two triangular spherical pyramids are to each other as their bases: and since a polygonal pyramid may always be divided into a certain number of triangular ones, it follows that any two spherical pyramids are to each other, as the polygons which form their bases.

Second. The solid angles at the vertices of those pyramids are also as their bases; hence, for comparing any two solid angles, we have merely to place their vertices at the centres of two equal spheres, and the solid angles will be to each other as the spherical polygons intercepted between their planes or faces.

The vertical angle of the tri-rectangular pyramid is formed by three planes at right angles to each other: this angle, which may be called a *right solid angle*, will serve as a very natural unit of measure for all other solid angles. And if so, the same number, that exhibits the area of a spherical polygon, will exhibit the measure of the corresponding solid angle. If the area of the polygon is $\frac{3}{4}$, for example, in other words, if the polygon is $\frac{3}{4}$ of the tri-rectangular polygon, then the corresponding solid angle will also be $\frac{3}{4}$ of the right solid angle.

PROPOSITION XXIV. THEOREM.

The surface of a spherical polygon is measured by the sum of all its angles, minus *the product of two right angles by the number of sides in the polygon* minus *two.*



From one of the vertices A, let diagonals AC, AD be drawn to all the other vertices; the polygon ABCDE will be divided into as many triangles *minus* two as it has sides. But the surface of each triangle is measured by the sum of all its angles *minus* two right angles; and the sum of the angles in all the triangles is evidently the same as that of all the angles in the polygon; hence the surface of the polygon is equal to the sum of all its angles diminished by twice as many right angles as it has sides *minus* two.

Scholium. Let s be the sum of all the angles in a spherical polygon, n the number of its sides; the right angle being taken for unity, the surface of the polygon will be measured by $s-2\ (n-2)$, or $s-2n+4$.

PROPOSITION XXV. THEOREM.*

Let S *be the number of solid angles in a polyedron,* H *the number of its faces,* A *the number of its edges; then in all cases, we shall have* $S+H=A+2$.

Within the polyedron, take a point, from which draw straight lines to the vertices of all its angles; conceive next, that from the same point as a centre, a spherical surface is described, meeting all these straight lines in as many points; join these points by arcs of great circles, so as to form on the surface of the sphere polygons corresponding in position and number with the faces of the polyedron. Let ABCDE (see the last figure) be one of these polygons, n the number of its sides; its surface will be $s-2n+4$, s being the sum of the angles A, B, C, D, E. If the surface of each polygon is estimated in a similar manner, and afterwards the whole are added together, we shall find their sum, or the surface of the sphere represented by 8, to be equal to the sum of all the angles in the polygons, *minus* twice the number of their sides, *plus* 4 taken as many times as there are faces. Now, since all the angles which lie round any one point A are equal to four right angles, the sum of all the angles in the polygons must be equal to 4 taken as many times as there are solid angles; it is therefore equal to $4S$. Also, twice the number of sides AB, BC, CD, &c. is equal to four times the number of edges, or to $4A$; because the same edge is always a side in two faces. Hence we have $8=4S-4A+4H$; or dividing all by 4, we have $2=S-A+H$; hence $S+H=A+2$.

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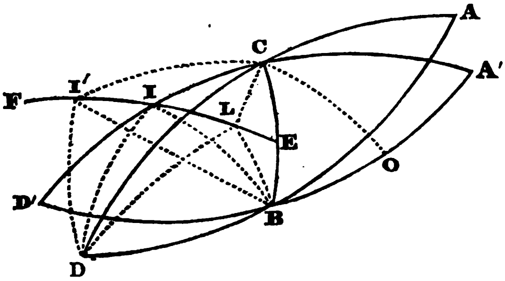
* This and the remaining Propositions of Book VII. may be omitted or not, as the reader thinks fit.—Ed.

Kor. From this it follows, that *the sum of all the plane angles, which form the solid angles of a polyedron, is equal to as many times four right angles as there are units in* $S-2$, S *being the number of solid angles in the polyedron.*

For, examining a face the number of whose sides is n, the sum of the angles in this face (25. I.) will be $2n-4$ right angles. But the sum of all these $2n$'s, or twice the number of sides in all the faces, will be $4A$; and 4 taken as many times as there are faces will be $4H$: hence the sum of the angles in all the faces is $4A-4H$. Now, by the Theorem just demonstrated, we have $A-H=S-2$, and consequently $4A-4H=4(S-2)$. Hence *the sum of all the plane angles, &c.*

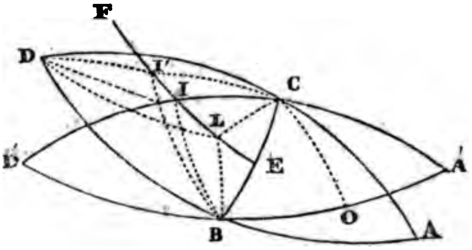
PROPOSITION XXVI. THEOREM.

Of all the spherical triangles formed with two given sides CA, CB, *and a third assumed at pleasure, the greatest* ABC *is the one in which the angle* C, *contained by the given sides, is equal to the sum of the two other angles* A *and* B.



Produce the two sides AC, AB till they meet in D; you will have a spherical triangle BCD, in which the angle DBC will also be equal to the sum of the two other angles BDC, BCD. For, BCD+BCA being equal to two right angles, and likewise CBA+CBD, we have BCD+BCA=CBA+CBD; and adding on both sides BDC=BAC, we shall have BCD+BCA+BDC=CBA+CBD+BAC. Now, by hypothesis, BCA=CBA+BAC; hence CBD=BCD+BDC.

Draw BI making the angle CBI=BCD, and consequently IBD=BDC; the two triangles IBD, IBC will be isosceles, and we shall have IC=IB=ID. Hence the point I, the middle point of BC, is at equal distances from the three points A, B, C.



Now, suppose $CA=CA$, and the angle $BCA'>BCA$; if $A'B$ be joined and the arcs $A'C$, $A'B$ produced till they meet in D', the arc $D'CA'$ will be a semicircumference, as well as DCA; therefore, since we have $CA'=CA$, we shall also have $CD'=CD$. But in the triangle CID', we have $CI+ID'>CD'$; hence $ID'>CD-CI$, or $ID'>ID$.

In the isosceles triangle CIB, bisect the angle I at the vertex, by the arc EIF, which will also bisect BC at right angles. If a point L is assumed between I and E, the distance BL, equal to LC, will be less than BI; for it might be shewn as in Prop. 9. I. that $BL+LC<BI+IC$; and taking the halves of each, that $BL>BI$. But in the triangle $D'LC$, we have $D'L>D'C-CL$, and still more $D'L>DC-CI$, or $D'L>DI$, or $D'L>BI$; hence $D'L>BL$. Hence if in the arc EIF, we seek for a point equally distant from the three points B, C, D', it can only be found in the prolongation of EI towards F. Let I be the point required, such that we have $D'I'=BI'=CI'$; the triangles $I'CB$, $I'CD'$, $I'BD'$ being isosceles, we shall have the equal angles $I'BC=I'CB$, $I'BD'=I'D'B$, $I'CD'=I'D'C$. But the angles $D'BC+CBA'$ are equal to two right angles, and likewise $D'CB+CBA'$ are equal to two right angles, and likewise $D'CB+BCA'$; hence

$$D'BI'+I'BC+CBA'=2,$$
$$BCI'-I'CD'+BCA'=2.$$

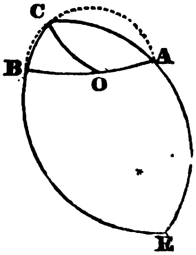
Add the two sums, observing that $I'BC=BCI'$, and $D'BI'-I'CD'=BD'I'-I'D'C=CD'B=CA'B$; we shall have

$$2I'BC+CA'B+CBA'+BCA'=4.$$

Hence $CA'B+CBA'+BCA'-2$ (which measures the area of the triangle $A'BC$)$=2-2I'BC$; so that we have *area* $A'BC=2-2$ *angle* $I'BC$; likewise, in the triangle ABC, we should have *area* $ABC=2-2$ *angle* IBC. Now the angle $I'BC$ has already been proved greater than IBC; hence the area $A'BC$ is less than ABC.

The same demonstration would lead to the same conclusion, if taking always the arc $CA'=CA$, the angle BCA' (see the fig. in the last page) were made greater than BCA; hence ABC is the greatest of all those triangles, having two sides given, and the third to be assumed at will.

Scholium 1. The triangle ABC, the greatest of all those which have two given sides CA, CB, may be inscribed in a semicircle, the diameter of which is the chord of the third side AB; for O being the middle point of AB, the distances OC, OB, as we have seen, are equal; hence the circumference of a little circle, described from the point O as a pole, with the distance OB, will pass through the three points A, B, C. Moreover, the straight line AB is a diameter of this little circle; for the centre, which must lie at once in the plane of the little circle, and (1. VII. Cor. 4.) in the plane of the arc of the great circle BOA, must of necessity be found in the intersection of those two planes, which is the straight line BA; hence BA will be a diameter.



Scholium 2. In the triangle ABC, the angle C being equal to the sum of the other two A and B, the sum of all the three angles must be double of the angle C. But (19. VII.) that sum is always greater than two right angles; hence C is always greater than one.

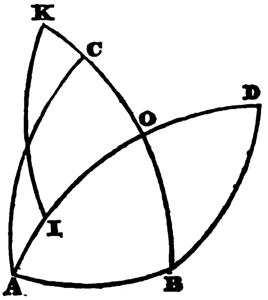
Scholium 3. If the sides CB, CA are produced till they meet in E, the triangle BAE will be equal to the fourth part of the surface of the sphere. For the angle $E=C=ABC+CAB$; hence the three angles of the triangle BAE are equivalent to the four ABC, ABE, CAB, BAE, whose sum is equal to four right angles; hence (24. VII.) the surface of the triangle $BAE=4-2=2$, which is the fourth part of the surface of the sphere.

Scholium 4. There could be no *maximum*, if the sum of the two given sides CA, CB were equal to, or greater than, the half-circumference of a great circle. For, since the triangle ABC must be capable of being inscribed in a semicircle of the sphere, the sum of the two sides CA, CB will be less (3. VII.) than the semicircumference BCA, and consequently less than half the circumference of a great circle.

The reason why there can be no *maximum*, when the sum of the two given sides is greater than the semicircumference of a great circle, is that in this case the triangle continues to augment, as the angle contained by its two given sides augments; and at last, when this angle becomes equal to two right angles, the three sides lie all in the same plane, and form a whole circumference; the spherical triangle has then increased to a hemisphere, but it has at the same time ceased to be a triangle.

PROPOSITION XXVII. THEOREM.

Of all the spherical triangles, formed with a given side and a given perimeter, the greatest is that in which the two undetermined sides are equal.



Let AB the given side be common to the two triangles ACB, ADB, and let AC+CB=AD+DB; we are to shew that the isosceles triangle ACB, in which AC=CB, is greater than ADB, which is not isosceles.

Since those triangles have the common part AOB, it will be enough to prove that the triangle BOD is less than AOC. Now, the angle CBA, equal to CAB, is greater than OAB; therefore (21. VII.) the side AO is greater than OB. Take OI=OB, make OK=OD, and join KI; the triangle OKI (21. VII.) will be equal to DOB. Now if the triangle DOB, or its equal KOI, is not admitted to be less than OAC, it must be either equal or greater; in both which cases, since the point I lies between A and O, the point K must be found in the prolongation of OC, otherwise the triangle OKI were contained in the triangle CAO, and therefore less than it. This granted, since the shortest path from C to A is CA, we have CK+KI+IA >CA. But CK=OD—CO, AI=AO—OB, KI=BD; hence OD—CO+AO—OB+BD>CA, or by reduction, AD—CB+BD>CA, or AD+BD>CA+CB. But this inequality is at variance with the supposition of AD+BD=CA+CB; hence the point K cannot fall in the prolongation of OC; hence it falls between O and C, and consequently the triangle KOI or its equal ODB is less than ACO; hence the isosceles triangle ACB is greater than ADB, which is not isosceles, and has the same base and perimeter.

Scholium. The last two Propositions are analogous to Prop. 1. and 3. of the Appendix to Book IV.; and from them may be deduced, in regard to spherical polygons, the same consequences as we found above to be true with regard to rectilineal polygons. The chief are as follows:

1. *Of all the isoperimetrical polygons having a given number of sides, the greatest is an equilateral polygon.*

Same demonstration as in Prop. 2. App. Book IV.

2. *Of all the spherical polygons, formed with sides all given except one, which may be assumed at pleasure, the greatest is that polygon which may be inscribed in a semicircle, having for its diameter the chord of the undetermined side.*

The demonstration is deduced from Prop. 26, in the manner exhibited in Prop. 4. of the Appendix just quoted. It is requisite for the existence of a *maximum*, that the sum of the given sides be less than the semicircumference of a great circle.

3. *The greatest of all the spherical polygons, formed with given sides, is that which can be inscribed in a circle of the sphere.*

Same demonstration as in Prop. 4. App. Book IV.

4. *The greatest of all the spherical polygons, having the same perimeter and the same number of sides, is that which has its angles equal and its sides equal.*

This results from the *first* and the *third* of these corollaries.

Note. All the propositions about *maxima* in spherical polygons, are, at the same time, applicable to solid angles, of which those polygons are the measures.

APPENDIX TO BOOKS VI. AND VII.

THE REGULAR POLYEDRONS.

PROPOSITION I. THEOREM.

There can only be five regular polyedrons.

For, *regular polyedrons* were defined as having equal regular polygons for their faces, and all their solid angles equal. These conditions cannot be fulfilled except in a small number of cases.

First. If the faces are equilateral triangles, polyedrons may be formed of them, having solid angles contained by three of those triangles, by four, or by five: hence arise three regular bodies, the *tetraedron*, the *octaedron*, the *icosaedron*. No other can be formed with equilateral triangles; for six angles of such a triangle are equal to four right angles, and (21. V.) cannot form a solid angle.

Secondly. If the faces are squares, their angles may be arranged by threes: hence results the *hexaedron* or *cube*. Four angles of a square are equal to four right angles, and cannot form a solid angle.

Thirdly. In fine, if the faces are regular pentagons, their angles likewise may be arranged by threes: the regular *dodecaedron* will result.

We can proceed no farther: three angles of a regular hexagon are equal to four right angles; three of a heptagon are greater.

Hence there can only be five regular polyedrons; three formed with equilateral triangles, one with squares, and one with pentagons.

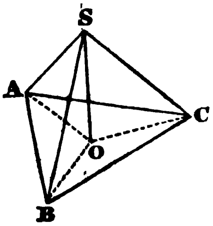
Scholium. In the following Proposition, we shall prove that these five polyedrons actually exist; and that all their dimensions may be determined, when one of their faces is known.

PROPOSITION II. PROBLEM.

One of the faces of a regular polyedron being given, or only a side of it, to construct the polyedron.

This Problem subdivides itself into five, which we shall now solve in succession.

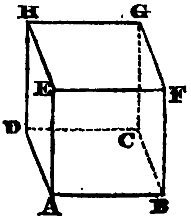
Construction of the Tetraedron.



Let ABC be the equilateral triangle which is to form one face of the tetraedron. At the point O, the centre of this triangle, erect OS perpendicular to the plane ABC; terminate this perpendicular in S, so that AS=AB; join SB, SC: the pyramid SABC will be the tetraedron required.

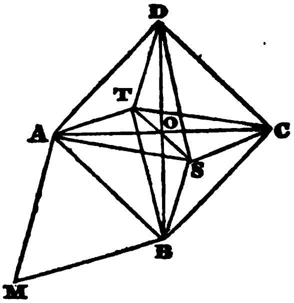
For, by reason of the equal distances OA, OB, OC, the oblique lines SA, SB, SC are equally removed from the perpendicular so, and consequently equal. One of them SA=AB; hence the four faces of the pyramid SABC are triangles, equal to the given triangle ABC. And the solid angles of this pyramid are all equal, because each of them is formed by three equal plane angles: hence this pyramid is a regular tetraedron.

Construction of the Hexaedron.



Let ABCD be a given square. On the base ABCD, construct a right prism whose altitude AE shall be equal to the side AB. The faces of this prism will evidently be equal squares; and its solid angles all equal, each being formed with three right angles: hence this prism is a regular hexaedron or cube.

Construction of the Octaedron.



Let AMB be a given equilateral triangle. On the side AB, describe a square ABCD; at the point O, the centre of this square, erect TS perpendicular to its plane, and terminating on both sides in T and S, so that OT=OS=OA; then join SA, SB, TA, &c.: you will have a solid SABCDT, composed of two quadrangular pyramids SABCD, TABCD, united together by their common base ABCD; this solid will be the required octaedron.

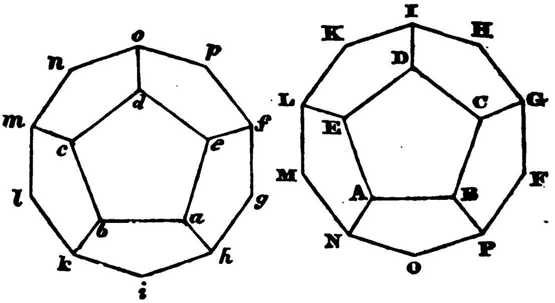
For, the triangle AOS is right-angled at O, and likewise the triangle AOD; the sides AO, OS, OD are equal; hence those triangles are equal, hence AS=AD. In the same manner we could shew, that, all the other right-angled triangles AOT, BOS, COT, &c. are equal to the triangle AOD; hence all the sides AB, AS, AT, &c. are equal, and therefore the solid SABCDT is contained by eight triangles, each equal to the given equilateral triangle ABM. We have yet to shew that the solid angles of this polyedron are equal to each other; that the angle S, for example, is equal to the angle B.

Now, the triangle SAC is evidently equal to the triangle DAC, and therefore the angle ASC is right; hence the figure SATC is a square equal to the square ABCD. But, comparing the pyramid BASCT with the pyramid SABCD, the base ASCT of the first may be placed on the base ABCD of the second; then, the point O being their common centre, the altitude OB of the first will coincide with the altitude OS of the second; and the two pyramids will exactly apply to each other in all points; hence the solid angle S is equal to the solid angle B; hence the solid SABCDT is a regular octaedron.

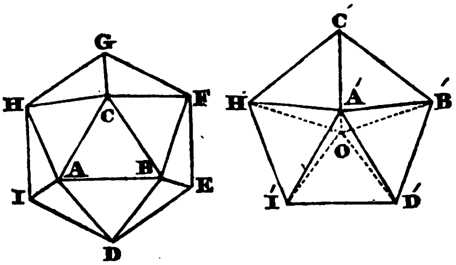
Scholium. If three equal straight lines AC, BD, ST are perpendicular to each other, and bisect each other, the extremities of these straight lines will be the vertices of a regular octaedron.

Construction of the Dodecaedron.

Let ABCDE be a given regular pentagon; let ABP, CBP be two plane angles each equal to the angle ABC. With these plane angles form the solid angle B; and by Prop. 24, V., determine the mutual inclination of two of those planes; which inclination we shall name K. In like manner, at the points C, D, E, A, form solid angles, equal to the solid angle B, and similarly situated: the plane CBP will be the same as the plane BCG, since both of them are inclined at an equal angle K to the plane ABCD: Hence in the plane PBCG, we may describe the pentagon BCGFP, equal to the pentagon ABCDE. If the same thing is done in each of the other planes CDI, DEL, &c., we shall have a convex surface PEGH, &c. composed of six regular pentagons, all equal, and each inclined to its adjacent plane by the same quantity K. Let *pfgh*, &c. be a second surface equal to PFGH, &c.; we assert that these two surfaces may be joined so as to form only a single continuous convex surface. For the angle *opf*, for example, may be joined to the two angles OPB, BPF, to make a solid angle P equal to the angle B; and in this junction, no change will take place in the inclination of the planes BPF, BPO, that inclination being already such as is required to form the solid angle. But whilst the solid angle P is forming, the side *pf* will apply itself to its equal PF, and at the point F will be found three plane angles PFG, *pfe*, *efg*, united to form a solid angle equal to each of the solid angles already formed: and this junction, like the former, will take place without producing any change either in the state of the angle P or in that of the surface *efgh*, &c.; for the planes PFG, *efp* already joined at P, have the requisite inclination K, as well as the planes *efg*, *efp*. Continuing the comparison, in this way, by successive steps, it appears that the two surfaces will adjust themselves completely to each other, and form a single continuous convex surface; which will be that of the regular dodecaedron, since it is composed of twelve equal regular pentagons, and has all its solid angles equal.



Construction of the Iosaedron.



Let ABC be one of its faces. We must first form a solid angle with five planes each equal to ABC, and each equally inclined to its adjacent one. To effect this, on the side B′C′, equal to BC, construct the regular pentagon B′C′H′I′D′; at the centre of this pentagon, draw a line at right angles to its plane, and terminating in A′, so that B′A′=B′C′; join A′C′, A′H′, A′I′, A′D′: the solid angle A′ formed by the five planes B′A′C′, C′A′H′, &c., will be the solid angle required. For the oblique lines A′B′, A′C′, &c. are equal; one of them A′B′ is equal to the side B′C′; hence all the triangles B′A′C′, C′A′H′, &c. are equal to each other and to the given triangle ABC.

It is farther manifest, that the planes B′A′C′, C′A′H′, &c. are each equally inclined to their adjacent planes; for the solid angles B′, C′, &c. are all equal, being each formed by two angles of equilateral triangles, and one of a regular pentagon. Let the inclination of the two planes, in which are the equal angles, be named K; which K may be determined by Prop. 24. V.; the angle K will at the same time be the inclination of each of the planes composing the solid angle A′ to their adjacent planes.

This being granted, if at each of the points A, B, C, a solid angle be formed equal to the angle A′, we shall have a convex surface DEFG, &c. composed of ten equilateral triangles, every one of which will be inclined to its adjacent triangle by the quantity K; and the angles D, E, F, &c. of its contour or rim will alternately combine three angles and two angles of equilateral triangles. Conceive a second surface equal to the surface DEFG, &c.; these two surfaces will adapt themselves to each other, if each triple angle of the one is joined to each double angle of the other; and, since the planes of these angles have already the mutual inclination K, requisite to form a quintuple solid angle equal to the angle A, there will be nothing changed by this junction in the state of either surface, and the two together will form a single continuous surface, composed of twenty equilateral triangles. This surface will be that of the regular icosaedron, since all its solid angles are likewise equal.

PROPOSITION III. PROBLEM.

To find the inclination of two adjacent faces of a regular polyedron.

This inclination is deduced immediately from the construction we have just given of the five regular polyedrons; taken in connexion with Prop. 24. V., by means of which, the three plane angles that form a solid angle being given, the angle which two of these plane angles form with each other may be determined.

In the tetraedron. Each solid angle is formed of three angles of equilateral triangles; therefore seek, by the Problem referred to, the angle which two of these planes contain between them: it will be the inclination of two adjacent faces of the tetraedron.

In the hexaedron. The angle contained by two adjacent faces is a right angle.

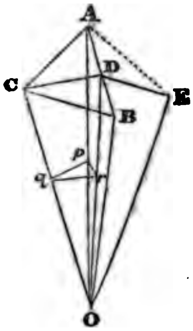
In the octaedron. Form a solid angle with two angles of equilateral triangles and a right angle: the inclination of the two planes, in which the triangular angles lie, will be that of two adjacent faces of the octaedron.

In the dodecaedron. Every solid angle is formed with three angles of regular pentagons: the inclination of the planes of two of these angles will be that of two adjacent faces of the dodecaedron.

In the icosaedron. Form a solid angle with two angles of equilateral triangles and one of a regular pentagon; the inclination of the two planes, in which the triangular angles lie, will be that of two adjacent faces of the icosaedron.

PROPOSITION IV. PROBLEM.

The side of a regular polyedron being given, to find the radii of the spheres, inscribed in this polyedron and circumscribing it.



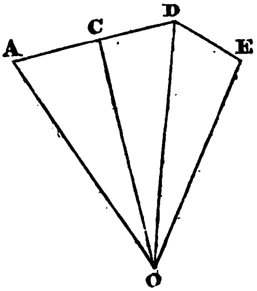
It must first be shewn, that every regular polyedron is capable of being inscribed in a sphere, and of circumscribing it.

Let AB be the side common to two adjacent faces; C and E the centres of those faces; CD, ED the perpendiculars let fall from these centres upon the common side AB, and therefore terminating in D the middle point of that side. The two perpendiculars CD, DE make with each other an angle which is known, being the inclination of two adjacent faces, and determinable by the last Problem. Now, if in the plane CDE, at right angles to AB, two indefinite lines CO and OE be drawn perpendicular to CD and ED, and meeting each other in O; this point O will be the centre of the inscribed and of the circumscribed sphere, the radius of the first being OC, that of the second OA.

For, since the apothems CD, DE are equal, and the hypotenuse DO is common, the right-angled triangle CDO must (18. I.) be equal to the right-angled triangle ODE, and the perpendicular OC to OE. But, AB being perpendicular to the plane CDE, the plane ABC (17. V.) is perpendicular to CDE, or CDE to ABC; likewise CO, in the plane CDE is perpendicular to CD, the common intersection of the planes CDE, ABC; hence (18. V.) CO is perpendicular to the plane ABC. For the same reason, EO is perpendicular to the plane ABE: hence the two straight lines CO, EO, drawn perpendicular to the planes of two adjacent faces, through the centres of those faces, will meet in the same point O, and be equal to each other. Now, suppose that ABC and ABE represent any other two adjacent faces; the apothem CD will still continue of the same magnitude; and also the angle CDO, the half of CDE: hence the right-angled triangle CDO, and its side CO will be equal in all the faces of the polyedron; hence, if from the point O as a centre with the radius OC, a sphere be described, it will touch all the faces of the polyedron at their centres, the planes ABC, ABE, &c. being each perpendicular to a radius at its extremity: hence the sphere will be inscribed in the polyedron, or the polyedron circumscribed about the sphere.

Again, join OA, OB: by reason of CA=CB, the two oblique lines OA, OB, lying equally remote from the perpendicular, will be equal; so also will any other two lines drawn from the centre O to the extremities of any one side: hence all those lines will be equal; hence, if from the point O as a centre, with the radius OA, a spherical surface be described, it will pass through the vertices of all the solid angles of the polyedron; hence the sphere will be circumscribed about the polyedron, or the polyedron inscribed in the sphere.

This being settled, the solution of our Problem presents no farther difficulty, and may be effected thus:



One face of the polyedron being given, describe that face; and let CD be its apothem. Find, by the last Problem, the inclination of two adjacent faces of the polyedron, and make the angle CDE equal to this inclination; take DE=CD; draw CO and EO perpendicular to CD and ED, respectively: these two perpendiculars will meet in a point O; and CO will be the radius of the sphere inscribed in the polyedron.

On the prolongation of DC, take CA equal to a radius of the circle, which circumscribes a face of the polyedron; AO will be the radius of the sphere circumscribed about this same polyedron.

For, the right-angled triangles CDO, CAO, in the present diagram, are equal to the triangles of the same name in the preceding diagram: and thus, while CD and CA are the radii of the inscribed and the circumscribed circles belonging to any one face of the polyedron, OC and OA are the radii of the inscribed and the circumscribed spheres which belong to the polyedron itself.

Scholium. From the foregoing Propositions, several consequences may be deduced.

1. Any regular polyedron may be divided into as many regular pyramids as the polyedron has faces; the common vertex of these pyramids will be the centre of the polyedron; and at the same time, that of the inscribed and of the circumscribed sphere.

2. The solidity of a regular polyedron is equal to its surface multiplied by a third part of the radius of the inscribed sphere.

3. Two regular polyedrons of the same name are two similar solids, and their homologous dimensions are proportional; hence the radii

of the inscribed or of the circumscribed spheres are to each other as the sides of the polyedrons.

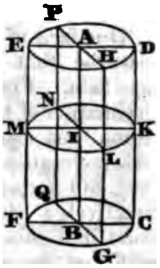
4. If a regular polyedron is inscribed in a sphere, the planes drawn from the centre, along the different edges, will divide the surface of the sphere into as many spherical polygons, all equal and similar as the polyedron has faces.

BOOK VIII.

THE THREE ROUND BODIES.

Definitions.

I. A *cylinder* is the solid produced by the revolution of a rectangle ABCD, conceived to turn about the immoveable side AB.



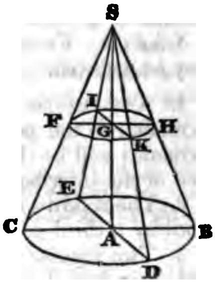
In this movement, the sides AD, BC, continuing always perpendicular to AB, describe equal circular planes DHP, CGQ, which are called the *bases of the cylinder*, the side CD at the same time describing the *convex surface.*

The immoveable line AB is called the *axis of the cylinder.*

Every section KLM, made in the cylinder, at right angles to the axis, is a circle equal to either of the bases; for, whilst the rectangle ABCD turns about AB, the line KI, perpendicular to AB, describes a circular plane, equal to the base, and this plane is nothing else than the section made perpendicular to the axis at the point I.

Every section PQGH, made along the axis, is a rectangle double of the generating rectangle ABCD.

II. A *cone* is the solid produced by the revolution of a right-angled triangle SAB, conceived to turn about the immoveable side SA.



In this movement, the side AB describes a circular plane BDCE, named the *base of the cone;* the hypotenuse SB describes its *convex surface.*

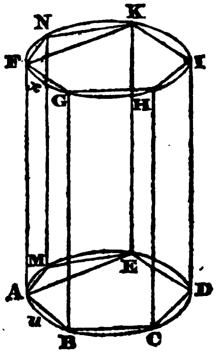
The point S is named the *vertex of the cone,* SA the *axis* or the *altitude,* and SB the *side* or the *apothem.*

Every section HKFI, formed at right angles to the axis, is a circle; every section SDE, formed along the axis, is an isosceles triangle double of the generating triangle SAB.

III. If from the cone SCDB, the cone SFKH be cut off by a section parallel to the base, the remaining solid CBHF is called a *truncated cone*, or the *frustum of a cone.*

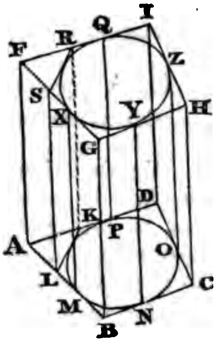
We may conceive it to be described by the revolution of a trapezium ABHG, whose angles A and C are right, about the side AG. The immoveable line AG is called the *axis* or *altitude of the frustum*, the circles BDC, HFK, are its *bases*, and BH is its *side.*

IV. Two cylinders, or two cones, are *similar*, when their axes are to each other as the diameters of their bases.



V. If in the circle ACD, which forms the base of a cylinder, a polygon ABCDE is inscribed, a right prism, constructed on this base ABCDE, and equal in altitude to the cylinder, is said to be *inscribed in the cylinder*, or the cylinder to be *circumscribed about the prism.*

The edges AF, BG, CH, &c. of the prism, being perpendicular to the plane of the base, are evidently included in the convex surface of the cylinder; hence the prism and the cylinder touch one another along these edges.



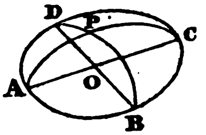
VI. In like manner, if ABCD is a polygon, circumscribed about the base of a cylinder, a right prism, constructed on this base ABCD, and equal in altitude to the cylinder, is said to be *circumscribed about the cylinder*, or the cylinder to be *inscribed in the prism.*

Let M, N, &c. be the points of contact in the sides AB, BC, &c.; and through the points M, N, &c. let MX, NY, &c. be drawn perpendicular to the plane of the base: those perpendiculars will evidently lie both in the surface of the cylinder, and in that of the circumscribed prism; hence they will be their lines of contact.

Note. The Cylinder, the Cone, and the Sphere, are the *three round bodies* treated of in the Elements of Geometry.

PRELIMINARY LEMMAS CONCERNING SURFACES.

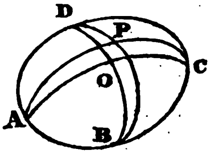
I. *A plane surface* OABDC *is less than any other surface* PABCD, *terminated by the same contour* ABCD.



This proposition is almost evident enough to be ranked in the class of axioms; for the plane may be regarded among surfaces, as being what the straight line is among lines; the straight line is the shortest distance between two given points; and so also, it may easily be conceived, is the plane the least of all the surfaces having the same perimeter. Yet, since it is advisable to reduce the number of axioms as far as possible, we have subjoined a demonstration, which will remove all doubt concerning this truth.

A surface being extended in length and breadth, one surface cannot be imagined to be greater than another, unless the dimensions of the first, in some direction, exceed those of the second: and if it should happen that the dimensions of one surface were, in all directions, less than the dimensions of another, the first surface would evidently be the less of the two. Now, in whatever direction we stretch the plane BPD to cut the plane surface along BD, and the other surface along BPD, the straight line BD will always be less than BPD; hence the plane surface OABCD is less than the surface PABCD, which envelopes it.

II. *Every* convex *surface* OABCD, *is less than any other surface enveloping it, and resting on the same contour* ABCD.



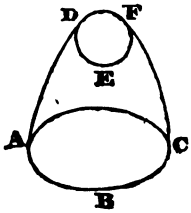
We shall here repeat, that, by *convex surface*, we understand a surface which cannot be cut by a straight line in more than two points: a straight line, however, may in some cases be capable of applying itself exactly in a certain direction to a convex surface; examples of this are to be seen in the surfaces of the cone and the cylinder. We may farther observe, that the name *convex surface*, is not limited to curve surfaces alone; it includes *polyedral* surfaces, or surfaces composed of several planes, and likewise surfaces partly curve and partly polyedral.

This being settled, if the surface OABCD, is not less than all those which envelope it, there must be among the latter a surface PABCD, less than all the rest, and at most, equal to PABCD. Through any point O, extend a plane, touching the surface OABCD, without cutting it; this plane will meet the surface PABCD, and (Lemma 1. VIII.) the part which it cuts off from that surface will be greater than the plane which terminates in the same boundary: hence, retaining the rest of the surface PABCD, we might substitute the plane instead of the part cut off from it, and so have a new surface, still enveloping OABCD, and less than PABCD.

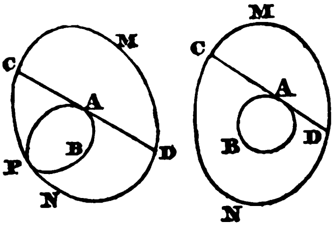
But by hypothesis, PABCD is the least of all; hence the hypothesis was false; hence the convex surface OABCD is less than any other surface enveloping it, and terminating in the same contour ABCD.

Scholium. By a mode of reasoning entirely similar, we could shew,

1. That, if a convex surface terminated by two contours ABC, DEF, is enveloped by any other surface, terminated by the same contours, the enveloped surface will be the smaller of the two.



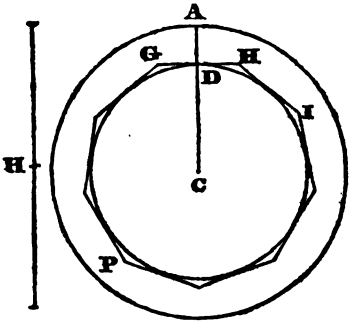
2. That, if a convex surface AB, is enveloped on all sides by another surface MN, whether they have any points, lines, or planes, in common, or have no point at all in common, the enveloped surface will always be less than the surface which envelopes it.



For, among the enveloping surfaces, there cannot be any one less than all the rest: because in every case a plane CD may be drawn so as to touch the enveloped convex surface, and (Lemma 1. VIII.) this plane will always be less than the surface CMD; whence the surface CND would be less than MN; which is contrary to the supposition of MN being the least of all. Hence the convex surface AB is less than all those which envelope it.

PROPOSITION I. THEOREM.

The solidity of a cylinder is equal to the product of its base by its altitude.



Let CA be a radius of the given cylinder's base; H the altitude; let *surf.* CA, represent the area of the circle whose radius is CA: we are to shew that the solidity of the cylinder is *surf.* CA × H. For, if *surf.* CA×H is not the measure of the given cylinder, it must be the measure of a greater cylinder, or of a smaller one. Suppose it first to be the measure of a smaller one; of a cylinder, for example, which has CD for the radius of its base, H being the altitude.

About the circle whose radius is CD, circumscribe a regular polygon GHIP (10. IV.), the sides of which shall not meet the circumference whose radius is CA. Imagine a right prism having the regular polygon GHIP for its base, and H for its altitude; this prism will be circumscribed about the cylinder, whose base has CD for its radius. Now, (14. VII.) the solidity of the prism is equal to its base GHIP, multiplied by the altitude H; the base GHIP is less than the circle, whose radius is CA; hence the solidity of the prism is less than *surf.* CA × H. But by hypothesis, *surf.* CA × H is the solidity of the cylinder inscribed in the prism; hence the prism must be less than the cylinder: whereas in reality it is greater, because it contains the cylinder; hence it is impossible that *surf.* CA × H can be the measure of the cylinder whose base has CD for its radius, H being the altitude; or in more general terms, *the product of the base, by the altitude of a cylinder, cannot measure a less cylinder.*

We must now prove that the same product cannot measure a greater cylinder. To avoid the necessity of changing our figure, let CD be a radius of the given cylinder's base; and, if possible, let *surf.* CD × H, be the measure of a greater cylinder, for example, of the cylinder, whose base has CA for its radius, H being the altitude.

The same construction being performed as in the first case, the prism, circumscribed about the given cylinder, will have GHIP×H for its measure: the area GHIP is greater than *surf.* CD; hence the solidity of this prism is greater than *surf.* CD×H: hence the prism must be greater than the cylinder having the same altitude, and *surf.* CA for its base. But on the contrary, the prism is less than the cylinder, being contained in it; hence *the base of a cylinder, multiplied by its altitude, cannot be the measure of a greater cylinder.*

Hence finally, the solidity of a cylinder is equal to the product of its base by its altitude.

Cor. 1. Cylinders of the same altitude are to each other as their bases; and cylinders of the same base are to each other as their altitudes.

Cor. 2. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases. For the bases are as the squares of their diameters; and the cylinders being similar, the diameters of their bases (*Def.* 4) are to each other as the altitudes: hence the bases are as the squares of the altitudes; hence the bases, multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

Scholium. Let R be the radius of a cylinder's base; H the altitude: the surface of the base (12. IV.) will be πR^2; and the solidity of the cylinder will be $\pi R^2 \times H$, or $\pi R^2 H$.

PROPOSITION II. LEMMA.

The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.

For this surface is equal to the sum of the rectangles AFGB, BGHC, CHID, &c. (see fig. of Def. 5.) which compose it. Now the altitudes AF, BG, CH, &c. of those rectangles, are equal to the altitude of the prism; their bases AB, BC, CD, &c. taken together, make up the perimeter of the prism's base. Hence the sum of these rectangles, or the convex surface of the prism, is equal to the perimeter of its base, multiplied by its altitude.

Cor. If two right prisms have the same altitude, their convex surfaces will be to each other as the perimeters of their bases.

PROPOSITION III. LEMMA.

The convex surface of a cylinder is greater than the convex surface of any inscribed prism, and less than the convex surface of any circumscribed prism.

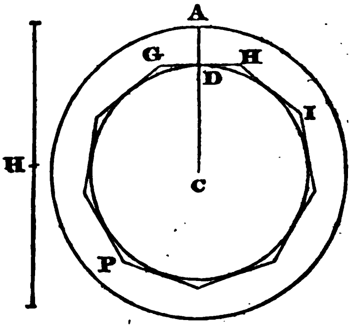
For (see the fig. of Def. 5.), the convex surface of the cylinder and that of the prism may be considered as having the same length, since every section made in either parallel to AF is equal to AF; and if these surfaces be cut, in order to obtain the breadths of them, by planes parallel to the base, or perpendicular to the edge AF, the one section will be equal to the circumference of the base, the other to the contour of the polygon ABCDE, which is less than that circumference: hence, with an equal length, the cylindrical surface is broader than the prismatic surface; hence the former is greater than the latter.

By a similar demonstration, the convex surface of the cylinder might be shewn to be less than that of any circumscribed prism BCDKLKH. (See the fig. of Def. 6.)

PROPOSITION IV. THEOREM.

The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.

Let CA be the radius of the given cylinder's base, H its altitude; the circumference whose radius is CA, being represented by *circ.* CA, we are to shew that *circ.* CA×H will be the convex surface of the cylinder. For, if this proposition is not true, then *circ.* CA×H must be the surface of a greater cylinder, or of a less one. Suppose it first to be the surface of a less cylinder; of the cylinder, for example, the radius of whose base is CD, and whose altitude is H.



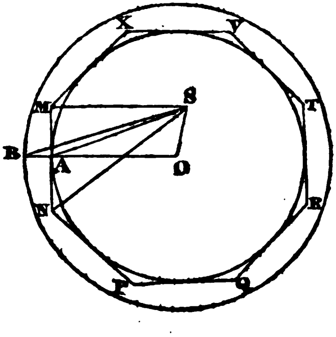
About the circle whose radius is CD, circumscribe a regular polygon GHIP, the sides of which shall not meet the circle whose radius is CA; conceive a right prism having H for its altitude, and the polygon GHIP for its base. The convex surface of this prism will be equal (2. VIII.) to the contour of the polygon GHIP multiplied by the altitude H: this contour is less than the circumference whose radius is CA; hence the convex surface of the prism is less than *circ.* CA × H. But, by hypothesis, *circ.* CA × H is the convex surface of the cylinder whose base has CD for its radius; which cylinder is inscribed in the prism: hence the convex surface of the prism must be less than that of the inscribed cylinder. On the other hand (3. VIII.) it is greater; hence our hypothesis was false: hence, in the first place, *the circumference of a cylinder's base multiplied by its altitude cannot be the measure of a smaller cylinder.*

We are next to shew that this product cannot be the measure of a greater cylinder. For, retaining the present figure, let CD be the radius of the given cylinder's base; and, if possible, let *circ.* CD×H be the convex surface of a cylinder, which with the same altitude has for its base a greater circle, the circle, for instance, whose radius is CA. The same construction being performed as above, the convex surface of the prism will again be equal to the contour of the polygon GHIP multiplied by the altitude H. But this contour is greater than *circ.* CD; hence the surface of the prism must be greater than *circ.* CD × H, which, by hypothesis, is the surface of cylinder having the same altitude, and CA for the radius of its base. Hence the surface of the prism must be greater than that of the prism. But even though this prism were inscribed in the cylinder, its surface (3. VIII.) would be less than the cylinder's; still farther is it less when the prism does not reach so far as to touch the cylinder. Hence our last hypothesis also was false; hence, in the second place, *the circumference of a cylinder's base multiplied by the altitude cannot measure the surface of a greater cylinder.*

Hence, finally, the convex surface of a cylinder is equal to the circumference of its base, multiplied by the altitude.

PROPOSITION V. THEOREM.

The solidity of a cone is equal to the product of its base by the third of its altitude.



Let SO be the altitude of the given cone, AO the radius of its base; the surface of the base being designated by *surf.* AO, we are to shew that *surf.* AO $\times\frac{1}{3}$SO is equal to the solidity of the cone.

Suppose, first, that *surf.* AO $\times\frac{1}{3}$SO, is the solidity of a greater cone; for example, of the cone whose altitude is still SO, but whose base has OB, greater than AO, for its radius.

About the circle whose radius is AO, circumscribe a regular polygon MNPT (10. IV.), so as not to meet the circumference whose radius is OB; imagine a pyramid having this polygon for its base, and the point S for its vertex. The solidity of this pyramid (19. VI.) is equal to the area of the polygon MNPT multiplied by a third of the altitude SO. But the polygon is greater than the inscribed circle represented by *surf.* AO; hence the pyramid is greater than *surf.* AO$\times\frac{1}{3}$SO, which, by hypothesis, measures the cone having S for its vertex and OB for the radius of its base: whereas, in reality, the pyramid is less than the cone, being contained in it; hence, first, the base of a cone multiplied by a third of its altitude cannot be the measure of a greater cone.

We are next to shew that this same product cannot be the measure of a smaller cone. For, retaining the present figure, let OB be the radius of the given cone's base; and, if possible, let *surf.* OB$\times^{1}$SO be the solidity of the cone having SO for its altitude, and for its base the circle whose radius is AO. The same construction as above being gone through, the pyramid SMNPT will have for its measure the area MNPT multiplied by $\frac{1}{3}$ SO. But the area MNPT is less than *surf.* OB; hence the measure of the pyramid must be less than *surf.* OB$\times\frac{1}{3}$SO, and consequently it must be less than the cone whose altitude is SO and whose base has AO for its radius. But, on the contrary, the pyramid is greater than the cone, because the cone is contained in it; hence, in the second place, the base of a cone multiplied by a third of its altitude cannot be the measure of a smaller cone.

Hence, finally, the solidity of a cone is equal to the product of its base by the third of its altitude.

Cor. A cone is the third of a cylinder having the same base and the same altitude; whence it follows,

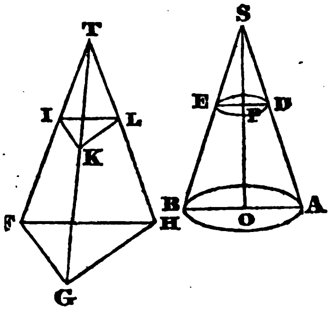
1. That cones of equal altitudes are to each other as their bases;
2. That cones of equal bases are to each other as their altitudes;
3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

Scholium. Let R be the radius of a cone's base, H its altitude; the solidity of the cone will be $\pi R^2 \times \frac{1}{3}H$, or $\frac{1}{3}\pi R^2 H$.

PROPOSITION VI. THEOREM.

The truncated cone ADEB, *whose altitude is* PO, *and the radii of whose bases are* AO, DP, *is measured by*

$$\tfrac{1}{3}\pi.OP.(AO^2+DP^2+AO\times DP).$$

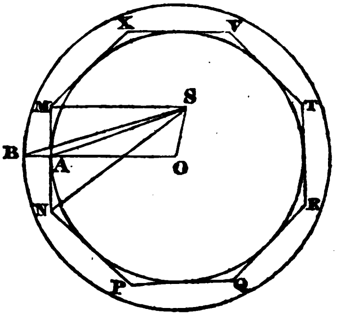


Let TFGH be a triangular pyramid having the same altitude as the cone SAB, and a base FGH equivalent to the base of the cone. These two bases may be conceived as placed on the same plane; in which case, the vertices S and T will be equally distant from the plane of the bases, and the plane EPD produced will form in the pyramid a section IKL. Now this section IKL is equivalent to the base DE: for (11. IV.) the bases AB, DE are to each other as the squares of the radii AO, DP, or as the squares of the altitudes SO, SP; the triangles FGH, IKL (15. VI.) are to each other as the squares of these same altitudes; hence the circles AB, DE are to each other as the triangles FGH, IKL. But by hypothesis, the triangle FGH is equivalent to the circle AB; hence the triangle IKL is equivalent to the circle DE.

Now, the base AB multiplied by $\frac{1}{3}$SO is the solidity of the cone SAB; and the base FGH multiplied by $\frac{1}{3}$SO is the solidity of the pyramid TFGH; hence, by reason of the equivalent bases, the solidity of the pyramid is equal to that of the cone. For a like reason, the pyramid TIKL is equivalent to the cone SDE; hence the truncated cone ADEB is equivalent to the truncated pyramid FGHIKL. But the base FGH, equivalent to the circle whose radius is AO, has for its measure $\pi\times AO^2$; likewise the base IKL has for its measure $\pi\times DP^2$; and the mean proportional between $\pi\times AO^2$ and $\pi\times DP^2$ is $\pi\times AO\times DP$; hence (20. VI.) the solidity of the truncated pyramid, or of the truncated cone, is measured by $\frac{1}{3}OP\times(\pi\times AO^2+\pi\times DP^2+\pi\times AO\times DP)$, or, which is the same thing, by $\frac{1}{3}\pi\times OP\times(AO^2+DP^2+AO\times DP)$.

PROPOSITION VII. THEOREM.

The convex surface of a cone is equal to the circumference of its base multiplied by half its side.



Let AO be a radius of the given cone's base, S its vertex, and SA its side: the surface will be *circ.* $AO\times\frac{1}{2}SA$. For, if possible, let *circ.* $AO\times SO$ be the surface of a cone having S for its vertex, and for its base a circle whose radius OB is greater than AO.

About the smaller circle describe a regular polygon MNPT, the sides of which shall not meet the circle whose radius is OB; and let SMNPT be the regular pyramid, having this polygon for its base and the point S for its vertex. The triangle SMN, one of those which compose the convex surface of the pyramid, has for measure its base MN multiplied by half its altitude SA, or half the side of the given cone; and since this altitude is the same in all the other triangles SNP, SPQ, &c., the convex surface of the pyramid must be equal to the perimeter MNPTM multiplied by $\frac{1}{2}$SA. But the contour MNPTM is greater than *circ.* AO; hence the convex surface of the pyramid is greater than *circ.* $AO\times\frac{1}{2}SA$, and consequently greater than the convex surface of the cone having the same vertex S and the circle whose radius is OB for its base. On the contrary, however, the surface of this cone is greater than that of the pyramid; because, if two such pyramids are adjusted to each other base to base, and two such cones base to base, the surface of the double cone will envelope on all sides that of the double pyramid, and therefore (Lemma 2.) be greater than it; hence the surface of the cone is greater than that of the pyramid. The reverse of this resulted from our hypothesis; hence that hypothesis was false; hence, in the first place, the circumference of the cone's base multiplied by half the side cannot measure the surface of a greater cone.

We are next to shew that it cannot measure the surface of a smaller cone. Let BO be the radius of the given cone's base; and, if possible, let *circ.* BO $\times \frac{1}{2}$SB be the surface of a cone having S for its vertex, and AO less than OB for the radius of its base.

The same construction being made as above, the surface of the pyramid SMNPT will still be equal to the perimeter MNPT $\times \frac{1}{2}$SA. Now this perimeter MNPT is less than *circ.* OB; likewise SA is less than SB: hence, for a double reason, the convex surface of the pyramid is less than *circ.* OB $\times \frac{1}{2}$SB, which, by hypothesis, is the surface of the cone having SA for the radius of its base; hence the surface of the pyramid must be less than that of the inscribed cone. On the contrary, however, it is greater; for, adjusting two such pyramids to each other base to base, and two such cones base to base, the surface of the double pyramid will envelope that of the double cone, and (Lemma 2.) will be greater than it. Hence, in the second place, the circumference of the given cone's base multiplied by half the side cannot be the measure of the surface of a smaller cone.

Hence finally, the convex surface of a cone is equal to the circumference of its base multiplied by half its side.

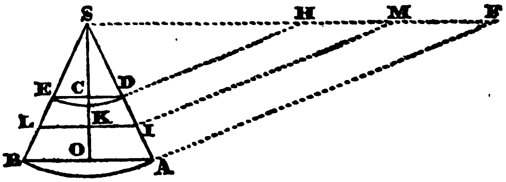
Scholium. Let L be the side of a cone, R the radius of its base; the circumference of this base will be 2πR, and the surface of the cone will be $2\pi\text{R} \times \frac{1}{2}\text{L}$, or πRL.

PROPOSITION VIII. THEOREM.

The convex surface of a truncated cone ADEB *is equal to its side* AD *multiplied by the half sum of* AB, DE, *the circumferences of its two bases.*

In the plane SAB which passes through the axis SO, draw the line AF perpendicular to SA, and equal to the circumference having AO for its radius; join SF; and draw DH parallel to AF.

By reason of the similar triangles SAO, SDC, we shall have AO : DC : : SA : SD; and by reason of the similar triangles



SAF, SDH, we shall have AF : DH : : SA : SD; hence AF : DH : : AO : DC, or (11. IV.) as *circ.* AO is to *circ.* DC. But by construction, AF=*circ.* AO; hence DH=*circ.* DC. This being granted, the triangle SAF, measured by AF × ½SA, is equal to the surface of the cone SAB which is measured by *circ.* AO×½SA. For a like reason, the triangle SDH is equal to the surface of the cone SDE. Hence the surface of the truncated cone ADEB is equal to that of the trapezium ADHF. The latter (7.III.) is measured by $\mathrm{AD}\times\left(\frac{\mathrm{AF}+\mathrm{DH}}{2}\right)$; hence the surface of the truncated cone ADEB is equal to its side AD multiplied by half the sum of the circumferences of its two bases.

Cor. Through I, the middle point of AD, draw IKL parallel to AB, and IM parallel to AF; it may be shewn as above that IM=*circ.* IK. But the trapezium ADHF=AD×IM=AD×*circ.* IK. Hence it may also be asserted, that *the surface of a truncated cone is equal to its side multiplied by the circumference of a section at equal distances from the two bases.*

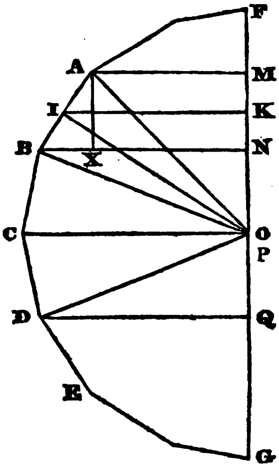
Scholium. If a line AD, lying wholly on one side of the line OC, and in the same plane, make a revolution around OC, the surface described by AD will have for its measure $\mathrm{AD}\times\left(\frac{circ.\ \mathrm{AO}+circ.\ \mathrm{DC}}{2}\right)$, or AD×*circ.* IK; the lines AO, DC, IK being perpendiculars, let fall from the extremities and from the middle of the axis OC.

For, if AD and OC are produced till they meet in S, the surface described by AD is evidently that of a truncated cone having AO and DC for the radii of its bases, the vertex of the whole cone being S. Hence this surface will be measured as we have said.

This measure will always hold good, even when the point D falls on S, and thus forms a whole cone; and also when the line AD is parallel to the axis, and thus forms a cylinder. In the first case DC would be nothing; in the second, DC would be equal to AO and to IK.

PROPOSITION IX. LEMMA.

Let AB, BC, CD *be several successive sides of a regular polygon,* O *its centre, and* OI *the radius of the inscribed circle; if that portion of the polygon* ABCD, *which lies wholly on one side of the diameter* FG, *be supposed to make a revolution about this diameter, the surface described by* ABCD *will have for its measure* MQ × circ. OI; MQ *being the altitude of that surface, or the axis included between* AM *and* DQ *the extreme perpendiculars.*

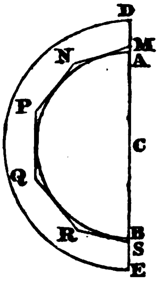


The point I being the middle of AB, and IK a perpendicular let fall from the point I upon the axis, the surface described by AB must (8. VIII.) have for its measure AB × *circ.* IK. Draw AX parallel to the axis; the triangles ABX, OIK will have their sides perpendicular each to each, namely, OI to AB, IK to AX, and OK to BX; hence these triangles are similar, and give the proportion AB : AX, or MN : : OI : IK, or as *circ.* OI to *circ.* IK; hence AB × *circ.* IK=MN × *circ.* OI. Whence it is plain that the surface described by the partial polygon ABCD is measured by (MN+NP+PQ) × *circ.* OI, or by MQ × *circ.* OI; hence it is equal to the altitude multiplied by the circumference of the inscribed circle.

Cor. If the whole polygon has an even number of sides, and if the axis FG passes through two opposite vertices F and G, the whole surface described by the revolution of the half-polygon FACG will be equal to its axis FG multiplied by the circumference of the inscribed circle. This axis FG will at the same time be the diameter of the circumscribed circle.

PROPOSITION X. THEOREM.

The surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.



We shall first shew, that the diameter of a sphere multiplied by the circumference of its great circle cannot measure the surface of a larger sphere. If possible, let AB × *circ.* AC be the surface of the sphere whose radius is CD.

About the circle whose radius is CA, circumscribe a regular polygon having an even number of sides, so as not to meet the circumference whose radius is CD: let M and S be the two opposite vertices of this polygon; and about the diameter MS let the half-polygon MPS be made to revolve. The surface described by this polygon will be measured (9. VIII.) by MS× *circ.* AC: but MS is greater than AB; hence the surface described by this polygon is greater than AB×*circ.* AC, and consequently greater than the surface of the sphere whose radius is CD. Now, on the contrary, the surface of the sphere is greater than the surface described by the polygon, since the former envelopes the latter on all sides. Hence, in the first place, the diameter of a sphere multiplied by the circumference of its great circle cannot measure the surface of a larger sphere.

We shall now shew that this same product cannot measure the surface of a smaller sphere. For, if possible, let DE×*circ.* CD be the surface of that sphere whose radius is CA. The same construction being made as in the former case, the surface of the solid generated by the revolution of the half-polygon will still be equal to MS×*circ.* AC. But MS is less than DE, and *circ.* AC is less than *circ.* CD; hence, for these two reasons, the surface of the solid described by the polygon must be less than DE×*circ.* CD, and therefore less than the surface of the sphere whose radius is AC. Now, on the contrary, the surface described by the polygon is greater than the surface of the sphere whose radius is AC, because the former envelopes the latter; hence, in the second place, the diameter of a sphere multiplied by the circumference of its great circle cannot measure the surface of a smaller sphere.

Hence the surface of a sphere is equal to its diameter multiplied by the circumference of its great circle.

Cor. The surface of the great circle is measured by multiplying its circumference by half the radius, or by a fourth of the diameter; hence *the surface of a sphere is four times that of its great circle.*

Scholium. The surface of the sphere being thus determined, and compared with plane surfaces, it will be easy to find the absolute value of the various lunes and spherical triangles whose ratio to the surface of the whole sphere has been determined above.

First, the lune having A for its angle, is to the surface of the sphere (20. VII.) as the angle A is to four right angles, or as the arc of the great circle which measures the angle A is to the circumference of that great circle. But the surface of the sphere is equal to the same circumference multiplied by the diameter; hence the surface of the lune is equal to the arc, which measures the angle of that lune, multiplied by the diameter.

In the second place, every spherical triangle is equal to the lune whose angle is half the excess of its three angles above two right angles. Let P, Q, R, then, be the three arcs of a great circle which measure the three angles of the triangle; let C be the circumference of a great circle, D its diameter; the spherical triangle will be equivalent to the lune whose angle is measured by $\frac{P+Q+R-\frac{1}{2}C}{2}$; and consequently its surface will be $D\times\left(\frac{P+Q+R-\frac{1}{2}C}{2}\right)$.

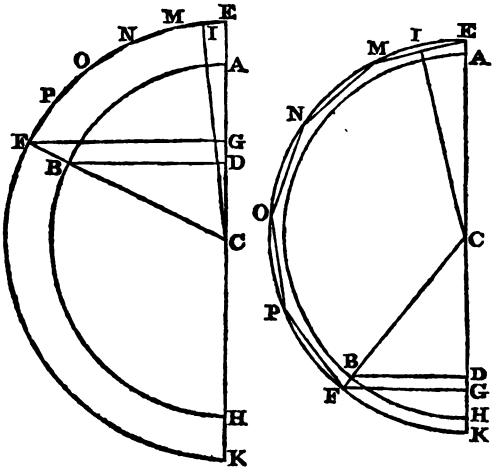
Thus in the case of the tri-rectangular triangle, each of the arcs P, Q, R being equal to $\frac{1}{4}C$, their sum will be $\frac{3}{4}C$, the excess of that sum above $\frac{1}{2}C$ will be $\frac{1}{4}C$, and half of that excess $\frac{1}{8}C$; hence the surface of the tri-rectangular triangle is $\frac{1}{8}C\times D$, the eighth part of the whole surface of the sphere.

The measurement of spherical polygons follows immediately from that of triangles; indeed it is entirely determined by Prop. 24. VII., since the unit of measure, or the tri-rectangular triangle, has now been estimated as a plane surface.

PROPOSITION XI. THEOREM.

The surface of any spherical zone is equal to its altitude multiplied by the circumference of a great circle.

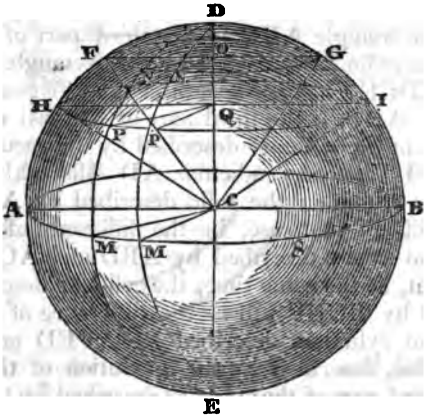
Let EF be any arc less or greater than a quadrant; and let FG be drawn perpendicular to the radius EC: the zone with one base, described by the revolution of the arc EF about EC, will be measured by EG × *circ.* EC.



For, suppose, first, that this zone is measured by something less; if possible, by EG × *circ.* CA. In the arc EF, inscribe a portion of a regular polygon EMNOPF, whose sides shall not reach the circumference described with the radius CA; and draw CI perpendicular to EM. By Prop. 9. VIII. the surface described by the polygon EMF turning about EC will be measured by EG × *circ.* CI. This quantity is greater than EG × *circ.* AC, which by hypothesis is the measure of the zone described by the arc EF. Hence the surface described by the polygon EMNOPF must be greater than the surface described by EF the circumscribed arc; whereas this latter surface is greater than the former, which it envelopes on all sides; hence, in the first place, the measure of any spherical zone with one base cannot be less than the altitude multiplied by the circumference of a great circle.

We are now to shew that the measure of this zone cannot be greater than its altitude multiplied by the circumference of a great circle. For suppose the zone described by the revolution of the arc AB about AC to be the proposed one; and, if possible, let *zone* AB $>$ AD × *circ.* AC. The whole surface of the sphere composed of the two zones AB, BH is measured by AH × *circ.* AC (10. VIII.), or by AD × *circ.* AC + DH × *circ.* AC; hence if we have *zone* AB $>$ AD × *circ.* AC, we must also have *zone* BH $>$ DH × *circ.* AC; which cannot be the case, as we have shewn above. Hence, in the second place, the measure of a spherical zone with one base cannot be greater than the altitude of this zone multiplied by the circumference of a great circle.

Hence, finally, every spherical zone with one base is measured by its altitude multiplied by the circumference of a great circle.



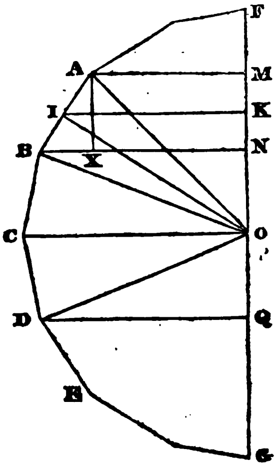
Let us now examine any zone with two bases, described by the revolution of the arc FH about the diameter DE. Draw FO, HQ perpendicular to this diameter. The zone described by the arc FH is the difference of the two zones described by the arcs DH and DF; the latter are respectively measured by DQ × *circ.* CD and DO × *circ.* CD; hence the zone described by FH has for its measure (DQ—DO) × *circ.* CD, or OQ × *circ.* CD.

Hence any spherical zone, with one or with two bases, is measured by its altitude multiplied by the circumference of a great circle.

Cor. Two zones, taken in the same sphere or in equal spheres, are to each other as their altitude; and any zone is to the surface of the sphere as the altitude of that zone is to the diameter.

PROPOSITION XIV. THEOREM.

Let AB, BC, CD *be several successive sides of a regular polygon,* O *its centre, and* OI *the radius of the inscribed circle; if the polygonal sector* AOD, *lying all on one side of the diameter* FG *be supposed to perform a revolution about this diameter, the solid so described will have for its measure* $\frac{1}{3}\pi . OI^2 . MQ$, MQ *being that portion of the axis which is included by the extreme perpendiculars* AM, DQ.

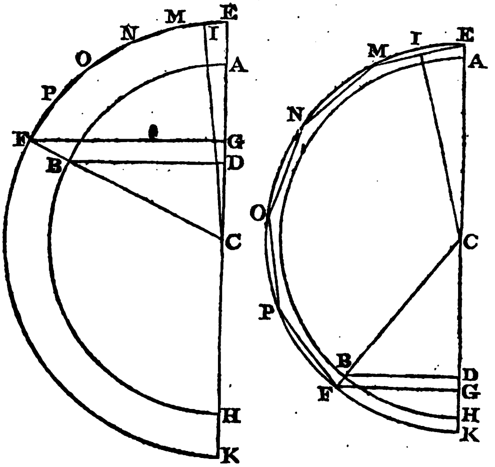


For, since the polygon is regular, all the triangles AOB, BOC, &c. are equal and isosceles. Now, by the last Corollary, the solid produced by the isosceles triangle AOB has for its measure $\frac{2}{3}\pi . OI^2 . MN$; the solid described by the triangle BOC has for its measure $\frac{2}{3}\pi . OI^2 . NP$; and the solid described by the triangle COD has for its measure $\frac{2}{3}\pi . OI^2 . PQ$: hence the sum of those solids, or the whole solid described by the polygonal sector AOD, will have for its measure $\frac{2}{3}\pi . OI^2 . (MN + NP + PQ$ or $\frac{2}{3}\pi OI^2 . MQ$.

PROPOSITION XV. THEOREM.

Every spherical sector is measured by the zone, which forms its base, multiplied by a third of the radius; and the whole sphere has for its measure a third of the radius, multiplied by its surface.

Let ABC be the circular sector, which, by its revolution about AC, describes the spherical sector: the zone described by AB being AD×*circ.* AC, or $2\pi . AC . AD$, we are to shew that this zone multiplied by $\frac{1}{3}$ of AC, or that $\frac{2}{3}\pi . AC^2 AD$, will measure the sector.



First, suppose, if possible, that $\frac{2}{3}\pi . AC^2 . AD$ is the measure of a greater spherical sector, say of the spherical sector described by the circular sector ECF similar to ACB.

In the arc EF, inscribe ECF, a portion of a regular polygon, such that its sides shall not meet the arc AB; then imagine the polygonal sector ENFC to turn about EC, at the same time with the circular sector ECF. Let CI be a radius of the circle inscribed in the polygon; and let FC be drawn perpendicular to EC. The solid described by the polygonal sector will (14. VIII.) have for its measure $\frac{2}{3}CI^2 . EG$; but CI is greater than AC by construction: and EG is greater thna AD; for joining AD, EF, the similar triangles EFG, ABD give the proportion EG : AD : : FG : BD : : CF : CB; hence $EG > AD$.

For this double reason, $\frac{2}{3}\pi CI^2 . EG$ is greater than $\frac{2}{3}\pi . CA^2 . AD$. The first is the measure of the solid described by the polygonal sector; the second, by hypothesis, is that of the spherical sector described by the circular sector ECF: hence the solid described by the polygonal sector must be greater than the spherical sector; whereas, in reality, it is less, being contained in the latter: hence our hypothesis was false; hence, in the first place, the zone or base of a spherical sector multiplied by a third of the radius, cannot measure a greater spherical sector.

We are next to shew that it cannot measure a less spherical sector. Let CEF be the circular sector, which, by its revolution, generates the given spherical sector; and suppose, if possible, that $\frac{2}{3}\pi.CE^2.EG$ is the measure of some smaller spherical sector, say of that produced by the circular sector ACB.

The construction remaining as above, the solid described by the polygonal sector will still have for its measure $\frac{2}{3}\pi.CI^2.EG$. But CI is less than CE; hence the solid is less than $\frac{2}{3}\pi.CE^2.EG$, which, according to our supposition, is the measure of the spherical sector described by the circular sector ACB. Hence the solid described by the polygonal sector must be less than the spherical sector described by ACB; whereas, in reality, it is greater, the latter being contained in the former. Hence, in the second place, it is impossible that the zone of a spherical sector, multiplied by a third of the radius, can be the measure of a smaller spherical sector.

Hence every spherical sector is measured by the zone, which forms its base, multiplied by a third of the radius.

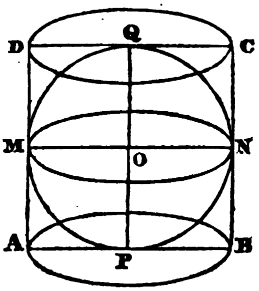
A circular sector ACB may increase till it becomes equal to a semicircle: in which case, the spherical sector described by its revolution is the whole sphere. Hence *the solidity of a sphere is equal to its surface multiplied by a third of the radius.*

Cor. The surfaces of spheres being as the squares of their radii, these surfaces multiplied by the squares of the radii must be as the cubes of the latter. Hence *the solidities of two spheres are as the cubes of their radii, or as the cubes of their diameters.*

Scholium. Let R be the radius of a sphere, its surface will be $4\pi R^2$; its solidity $4\pi R^2 \times \frac{1}{3}R$, or $\frac{4}{3}\pi.R^3$. If the diameter is named D, we shall have $R=\frac{1}{2}D$, and $R^3=\frac{1}{8}D^3$; hence the solidity may likewise be expressed by $\frac{4}{3}\pi \times \frac{1}{8}D^3$, or $\frac{1}{6}\pi D^3$.

PROPOSITION XVI. THEOREM.

The surface of a sphere is to the whole surface of the circumscribed cylinder (including its bases), as 2 is to 3. The solidities of these two bodies are to each other in the same ratio.



Let MNPQ be a great circle of the sphere; ABCD the circumscribed square: if the semicircle PMQ and the half square PADQ are at the same time made to revolve about the diameter PQ, the semicircle will generate the sphere, while the half-square will generate the cylinder circumscribed about that sphere.

The altitude AD of that cylinder is equal to the diameter PQ; the base of the cylinder is equal to the great circle, its diameter AB being equal to MN; hence (4. VIII.), the convex surface of the cylinder is equal to the circumference of the great circle multiplied by its diameter. This measure (10. VIII.) is the same as that of the surface of the sphere: hence *the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles; hence the convex surface of the cylinder is also equal to four great circles: and adding the two bases, each equal to a great circle, the total surface of the circumscribed cylinder will be equal to six great circles; hence the surface of the sphere is to the total surface of the circumscribed cylinder as 4 is to 6, or as 2 is to 3; which was the first branch of our Proposition.

In the next place, since the base of the circumscribed cylinder is equal to a great circle, and its altitude to the diameter, the solidity of the cylinder (1. VIII.) will be equal to a great circle multiplied by its diameter. But (15. VIII), the solidity of the sphere is equal to four great circles multiplied by a third of the radius, in other terms, to one great circle multiplied by $\frac{4}{3}$ of the radius, or by $\frac{2}{3}$ of the diameter; hence the sphere is to the circumscribed cylinder as 2 to 3, and consequently the solidities of these two bodies are as their surfaces.

Scholium. Conceive a polyedron, all of whose faces touch the sphere; this polyedron may be considered as formed of pyramids, each having for its vertex the centre of the sphere, and for its base one of the polyedron's faces. Now it is evident that all these pyramids will have the radius of the sphere for their common altitude; so that each pyramid will be equal to one face of the polyedron multiplied by a third of the radius:

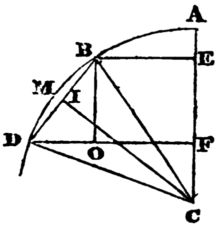
hence the whole polyedron will be equal to its surface multiplied by a third of the inscribed sphere's radius.

It is therefore manifest, that, the solidities of polyedrons circumscribed about the sphere are to each other as the surfaces of those polyedrons. Thus the property, which we have shewn to be true with regard to the circumscribed cylinder, is also true with regard to an infinite number of other bodies.

We might likewise have observed, that the surfaces of polygons, circumscribed about the circle, are to each other as their perimeters.

PROPOSITION XVII. PROBLEM.

The circular segment BMD *being supposed to make a revolution about a diameter exterior to it, to find the value of the solid so produced.*



On the axis, let fall the perpendiculars BE, DF; from the centre C, draw CI perpendicular to the chord BD; also draw the radii CB, CD.

The solid described by the sector BCA is equal to $\frac{2}{3}\pi . CB.^2AE$ (15. VIII.); the solid described by the sector $DCA = \frac{2}{3}\pi . CB^2 . AF$; hence the difference of these two solids, or the solid described by the sector $DCB = \frac{2}{3}\pi . CB^2 . (AF - AE) = \frac{2}{3}\pi . CB^2 . EF$. But the solid described by the isosceles triangle DCB (14. VIII.) has for its measure $\frac{2}{3}\pi . CI^2 . EF$; hence the solid described by the segment $BMD = \frac{2}{3}\pi . EF . (CB^2 - CI^2)$. Now, in the right-angled triangle CBI, we have $CB^2 - CI^2 = BI^2 = \frac{1}{4}BD^2$; hence the solid described by the segment BMD will have for its measure $\frac{2}{3}\pi . EF . \frac{1}{4}BD^2$, or $\frac{1}{6}\pi . BD^2 . EF$.

Scholium. The solid described by the segment BMD is to the sphere, which has BD for its diameter, as $\frac{1}{6}\pi . BD^2 . EF$ is to $\frac{1}{6}\pi . BD^3$, or as EF to BD.

PROPOSITION XVIII. THEOREM.

Every segment of a sphere, included between two parallel planes, is measured by the half-sum of its bases multiplied by its altitude, plus *the solidity of a sphere whose diameter is that same altitude.*

Let BE, DF (see the preceding figure), be the radii of the segment's two bases, EF its altitude, our segment thus being the one produced by the revolution of the circular space BMDFE about the axis FE. The solid decribed by the segment BMD is equal to $1\pi . BD^2 . EF$ (17. VIII.); and (6. VIII.) the truncated cone described by the trapezium BDFE is equal to $\frac{1}{3}\pi.\mathrm{EF}.(\mathrm{BE}^2+\mathrm{DF}^2+\mathrm{BE}.\mathrm{DF})$; hence the segment of the sphere, which is the sum of those two solids, must be equal to $\frac{1}{6}\pi.\mathrm{EF}.(2\mathrm{BE}^2+2\mathrm{DF}^2+2\mathrm{BE}.\mathrm{BF}+\mathrm{BD}^2)$. But, drawing DO parallel to EF, we shall have $\mathrm{DO}=\mathrm{DF}-\mathrm{BE}$, hence (9. III.) $\mathrm{DO}^2=\mathrm{DF}^2-2\,\mathrm{DF}.\mathrm{BE}+\mathrm{BE}^2$; and consequently $\mathrm{BD}^2=\mathrm{BO}^2+\mathrm{DO}^2=\mathrm{EF}^2+\mathrm{DF}^2-2\,\mathrm{DF}.\mathrm{BE}+\mathrm{BE}^2$. Put this value in place of BD^2 in the expression for the value of the segment, omitting the parts which destroy each other; we shall obtain for the solidity of the segment,

$$\tfrac{1}{6}\pi\,\mathrm{EF}.\,(3\,\mathrm{BE}^2+3\,\mathrm{DF}^2+\mathrm{EF}^2),$$

an expression which may be decomposed into two parts; the one, $\frac{1}{6}\pi.\ \mathrm{EF}.\ (3\,\mathrm{BE}^2+3\,\mathrm{DF}^2)$, or $\mathrm{EF}.\left(\frac{\pi.\mathrm{BE}^2+\pi.\mathrm{DF}^2}{2}\right)$ being the half-sum of the bases multiplied by the altitude; while the other $\frac{1}{6}\pi.\ \mathrm{EF}^3$ represents (15. VIII. Schol.) the sphere of which EF is the diameter: hence every segment of a sphere, &c.

Cor. If either of the bases is nothing, the segment in question becomes a spherical segment with a single base; hence *any spherical segment, with a single base, is equivalent to half the cylinder having the same base and the same altitude,* plus *the sphere of which this altitude is the diameter.*

General Scholium.

Let R be the radius of a cylinder's base, H its altitude: the solidity of the cylinder will be $\pi\mathrm{R}^2\times\mathrm{H}$, or $\pi\mathrm{R}^2\mathrm{H}$.

Let R be the radius of a cone's base, H its altitude: the solidity of the cone will be $\pi\mathrm{R}^2\times\frac{1}{3}\mathrm{H}$, or $\frac{1}{3}\pi\mathrm{R}^2\mathrm{H}$.

Let A and B be the radii of the bases of a truncated cone, H its altitude: the solidity of the truncated cone will be $\frac{1}{3}\pi.\mathrm{H}.(\mathrm{A}^2+\mathrm{B}^2+\mathrm{AB})$.

Let R be the radius of a sphere; its solidity will be $\frac{4}{3}\pi\mathrm{R}^3$.

Let R be the radius of a spherical sector, H the altitude of the zone, which forms its base: the solidity of the sector will be $\frac{2}{3}\pi\mathrm{R}^2\mathrm{H}$.

Let P and Q be the two bases of a spherical segment, H its altitude: the solidity of the segment will be $\frac{\mathrm{P}+\mathrm{Q}}{2}.\ \mathrm{H}+\frac{1}{6}\pi\mathrm{H}^3$.

If the spherical segment has but one base, the other being nothing, its solidity will be $\frac{1}{2}\mathrm{PH}+\frac{1}{6}\pi\mathrm{H}^3$.

END OF THE ELEMENTS OF GEOMETRY.

NOTES

ON THE

ELEMENTS OF GEOMETRY.

NOTE I.

On some Names and Definitions.

Some new expressions and definitions have been employed in this Work, where they seemed likely to give more accuracy and precision to geometrical language. We mean here to give some account of those changes, and to propose a few others, which might accomplish the same purpose more completely.

In the ordinary definition of the *rectangular parallelogram* and of the *square*, it is usual to say, that the angles of those figures are right; it would be more correct to say, that their angles are equal. For, to suppose that the four angles of a quadrilateral can be right, and even that the right angles are equal to each other, is to assume two propositions which require demonstration. This inconvenience, and several others of the same sort, might be avoided, if, instead of placing the definitions, according to the common practice, at the head of each Book, we were to disperse them over the course of the Book, each at the place where all it assumes is already proved.

The word *parallelogram*, according to its etymology, signifies *parallel lines;* it no more suits the figure of four sides, than it does that of six, of eight, &c. which have their opposite sides parallel. In like manner, the word *parallelepipedon* signifies *parallel planes;* it no more designates the solid with six faces, than the solid with eight, ten, &c. of which the opposite faces are parallel. The names, parallelogram and parallelepipedon, have the additional inconvenience of being very long. Perhaps, therefore, it would be advantageous to banish them altogether from geometry; and to substitute in their stead, the names *rhombus* and *rhomboid,* retaining the term *lozenge,* for quadrilaterals whose sides are all equal.

It might also be useful to extend the meaning of the word *inclination,* so as to make it synonymous with *angle:* both of them indicate a particular relation of two lines, or of two planes, which meet together, or would meet if produced. The inclination of two lines is nothing, when their angle is nothing; in other words, when the lines coincide, or lie parallel to each other. The inclination is greater when the angle is greater, or when the two lines form together a very obtuse angle. The quality of *sloping* has a different meaning; a line *slopes* the more towards another, the more it deviates from the perpendicular to that other.

It is customary with Euclid, and various geometrical writers, to give the name *equal triangles*, to triangles which are equal only in surface; and of *equal solids*, to solids which are equal only in solidity. We have thought it more suitable to call such triangles or solids *equivalent;* reserving the denomination *equal triangles* or *solids* for such as coincide when applied to each other.

In solids and curve surfaces, it is farther necessary to distinguish two sorts of equality, which differ in some respects. Two solids, two solid angles, two spherical triangles or polygons, may be equal in all their constituent parts, and yet be incapable of coinciding when applied to each other,—an observation which seems to have escaped the notice of elementary writers, as their inattention to it has vitiated certain demonstrations relying on the coincidence of figures, where no such coincidence can exist. Such are the demonstrations by which the equality of spherical triangles is sometimes imagined to be shewn, in the same manner as that of rectilineal triangles which are similarly related. A striking example of this oversight is exhibited by Robert Simson,* when this geometer impugns the demonstration of Euclid's Prop. 28. XI., yet falls himself into the error of grounding his own demonstration upon a coincidence which cannot take place. For these reasons, we have judged it necessary to assign a particular name to this kind of equality, which is not accompanied by coincidence; we have called it *equality by symmetry*, the figures to which it applies being called *symmetrical*.

Thus the terms *equal* figures, *symmetrical* figures, *equivalent* figures, refer to different objects, and should not be confounded in the same denomination.

In those propositions which relate to polygons, solid angles, and polyedrons, we have formally excluded all figures that have re-entrant angles. Our reason was, that, besides the propriety of limiting an elementary work to the simplest cases, if this exclusion had not taken place, several propositions either would not have been true, or, at least, would have required some modifications. We thought it better to restrict our reasoning to those lines and surfaces which we have named *convex*, and which are such that a straight line cannot cut them in more than two points.

We have frequently employed the expression, *product of two or more lines;* by which is meant, the product of the numbers representing those lines, when valued according to a linear unit, assumed at will. The signification of the phrase once fixed, there can be no objection against using it. In the same manner must be understood what is meant by the product of a surface by a line, of a solid by a surface, &c. It is enough to have settled, once for all, that such products are, or ought to be, considered as products of numbers, each of the kind proper to it. Thus, the product of a surface by a solid; is nothing but the product of a number of superficial units by a number of solid units.

In ordinary language, the word *angle* is often employed to designate the point situated at the vertex. This expression is inaccurate. It would be more correct and precise to use a particular name, such as that of *vertices*, for designating the points at the corners of a polygon or of a polyedron. The denomination *vertices of a polyedron*, as employed by us, is to be understood in this sense.

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* See his work, entitled: *Euclidis Elementorum libri sex, &c. Glasguæ*, 1756.

We have followed the common definition of *similar rectilineal figures;* we must observe, however, that it contains three superfluous conditions. In order to construct a polyedron, the number of whose sides is n, it is necessary first to know one side, and, next, the position of the vertices of all the angles situated out of this side. Now, the number of those angles is $n-2$, and the position of each vertex requires two *data;* hence, the whole number of data requisite for constructing a regular polygon of n sides, is $1 + 2n - 4$, or $2n - 3$. But in the similar polygon, one side may be assumed at will, therefore the number of conditions regulating the similarity of a polygon to a given polygon, is $2n - 4$. Now the common definition requires, *first*, that the angles be equal each to each, which amounts to n conditions; *secondly*, that the homologous sides be proportional, which amounts to $n - 1$ conditions. Consequently, there are $2n - 1$ conditions in all, therefore three too many. To obviate this inconvenience, the definition might be subdivided into two, as follows:

First. *Two triangles are similar, when they have two angles in each respectively equal.*

Second. *Two polygons are similar, when both may be divided into the same number of triangles, similar each to each, and similarly placed.*

But to prevent this latter definition itself from including any superfluous conditions, the number of triangles must be fixed equal to the number of the polygon's sides, *minus* two; which may be accomplished in either of the following ways: From the vertices of two homologous angles, diagonals may be drawn to all the opposite angular points; in which case, all the triangles formed in each polygon will have a common vertex, and their sum will be equal to the polygon: Or, let all the triangles formed in one polygon, have a side of it for their common base, and for vertices, the vertices of the different angles opposite to this base. In both cases, the number of triangles formed in the respective polygons being $n-2$, the conditions of their similarity will amount to $n-4$; the definition will contain nothing superfluous whatever; and this being once settled, the old definition will become a theorem susceptible of immediate demonstration.

If the definition of similar rectilineal figures usually given in elementary works is imperfect, that of *similar solid polyedrons* is much more so. Euclid makes this definition to depend on a theorem which is not proved; in other treatises, it has the inconvenience of being very redundant. We have, therefore, rejected those definitions of similar solids, and substituted one in their place, which is founded on the principles just explained. But as many other observations upon this subject solicit our attention, we shall return to it in a separate Note.

Our definition of *the perpendicular to a plane* may be looked upon as a theorem; that of the *inclination of two planes* likewise requires to be sanctioned by a train of reasoning; several others do the same. Accordingly, while in conformity to custom, we have retained the old definitions, care has been taken to refer the reader to those Propositions where their accuracy is demonstrated; or, in other cases, to subjoin a brief explanation.

The *angle* formed by the meeting *of two planes*, and the *solid angle* formed by the meeting of several planes at the same point, are magnitudes, each of its own kind, to which it would perhaps be convenient to give separate names. As they stand at present, it is difficult to avoid obscurities or circumlocutions, when speaking of the arrangement of the planes which compose the surface of a polyedron. And as the theory of those solids has hitherto been little investigated, no great inconvenience could arise from introducing any new expressions which are called for by the nature of the objects.

The angle formed by two planes, I should therefore propose to denominate a *corner;* the *edge* or *ridge* of the corner might designate the common intersection of the two planes. The corner might be named by means of four letters, the middle two corresponding to the edge. A *right corner* would be the angle formed by two planes perpendicular to each other. Four right corners would fill up all the solid angular space about a given line. Under this new denomination, the corner would still, not the less, have for its measure, the angle formed by the two perpendiculars, drawn each in its own plane, at the same point, to the edge or common intersection.

NOTE II.

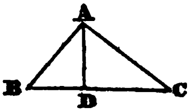
On the Demonstration of Prop. 20. Book I, *and of some other fundamental Propositions in Geometry.*

Prop. 20. I. is only a particular case of the famous *postulate*, on which Euclid has founded the doctrine of parallels, and likewise the theorem concerning the sum of the three angles of a triangle. This postulate has never hitherto been demonstrated in a way strictly geometrical, and independent of all considerations about infinity,—a circumstance attributable, doubtless, to the imperfection of our common definition of a straight line, on which the whole of geometry hinges. But viewing the matter in a more abstract light, we are furnished by analysis with a very simple method of rigorously proving both this and the other fundamental propositions of geometry. We here propose to explain this method, with all requisite minuteness, beginning with the theorem concerning the sum of the three angles of a triangle.

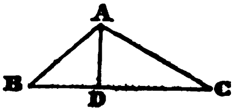
By superposition, it can be shewn immediately, and without any preliminary propositions, that *two triangles are equal when they have two angles and an interjacent side in each equal.* Let us call this side p, the two adjacent angles A and B, the third angle C. This third angle C, therefore, is entirely determined, when the angles A and B, with the side p, are known; for if several different angles C might correspond to the three given magnitudes A, B, p, there would be several different triangles, each having two angles and the interjacent side equal, which is impossible; hence the angle C must be a determinate function of the three quantities A, B, p, which I shall express thus, $C = \varphi : (A, B, p)$.

Let the right angle be equal to unity, then the angles A, B, C, will be numbers included between o and 2; and since $C = \varphi: (A, B, p)$, I assert, that the line p cannot enter into the funtion φ. For we have already seen that C must be entirely determined by the given quantities A, B, p alone, without any other line or angle whatever. But the line p is heterogeneous with the numbers A, B, C; and if there existed any equation between A, B, C, p, the value of p might be found from it in terms of A, B, C; whence it would follow, that p is equal to a number;' which is absurd: hence p cannot enter into the function φ, and we have simply $C = \varphi: (A, B)$.*

This formula already proves, that if two angles of one triangle are equal to two angles of another, the third angle of the former must also be equal to the third of the latter; and this granted, it is easy to arrive at the theorem we have in view.



First, let ABC be a triangle right-angled at A; from the point A, draw AD perpendicular to the hypotenuse. The angles B and D of the triangle ABD are equal to the angles B and A of the triangle BAC; hence, from what has just been proved, the third angle BAD is equal to the third C. For a like reason, the angle DAC = B, hence BAD + DAC, or BAC = B + C; but the angle BAC is right; hence *the two acute angles of a right-angled triangle are together equal to a right angle.*



Now, let BAC be any triangle, and BC a side of it not less than either of the other sides; if from the opposite angle A, the perpendicular AD is let fall on BC, this perpendicular will fall within the triangle ABC, and divide it into two right-angled triangles BAD, DAC. But in the right-angled triangle BAD, the two angles BAD, ABD are together equal to a right angle; in the right-angled triangle DAC, the two DAC, ACD are also equal to a right angle; hence all the four taken together, or, which amounts to the same thing, all the three BAC, ABC, ACB are together equal to two right angles; hence *in every triangle, the sum of its three angles is equal to two right angles.*

It thus appears, that the theorem in question does not depend, when considered *a priori*, upon any series of propositions, but may be deduced immediately from the principle of homogeneity; a principle which must display itself in a relations subsisting between all quantities of whatever sort. Let us continue the investigation, and shew that from the same source, the other fundamental theorems of geometry may likewise be derived.

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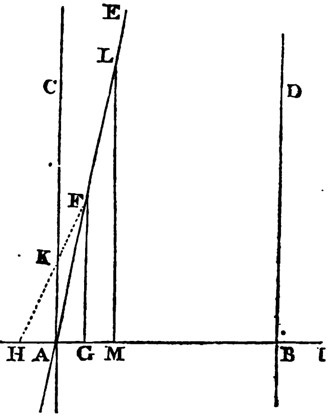
* Against this demonstration it has been objected, that if it were applied word for word to spherical triangles, we should find that two angles being known, are sufficient to determine the third, which is not the case in that species of triangles. The answer is, that in spherical triangles, there exists one element more than in plane triangles, the radius of the sphere, namely, which must not be omitted in our reasoning. Let r be the radius; instead of $C = \varphi (A, B, p)$, we shall now have $C = \varphi (A, B, p, r)$, or by the law of homogeneity, simply $C = \varphi \left(A, B, \frac{p}{r}\right)$. But since the ratio $\frac{p}{r}$ is a number, as well as A, B, C, there is nothing to hinder $\frac{p}{r}$ from entering the function φ, and consequently we have no right to infer from it, that $C = \varphi (A, B)$.

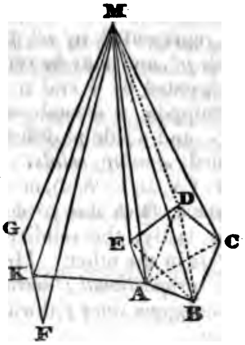
Retaining the same denominations as above, let us farther call the side opposite the angle A by the name of m, and the side opposite B by that of n. The quantity m must be entirely determined by the quantities A, B, p alone; hence m is a function of A, B, p, and $\frac{m}{p}$ is one also; so that we may put $\frac{m}{p} = \psi : (A, B, p)$. But $\frac{m}{p}$ is a number, as well as A and B; hence the function ψ cannot contain the line p, and we shall have simply $\frac{m}{p} = \psi : (A, B)$, or $m = p\psi : (A, B)$. Hence, also, in like manner, $n = p\,\psi\,(B, A)$.

Now, let another triangle be formed with the same angles A, B, C, and with sides m', n', p', respectively opposite to them. Since A and B are not changed, we shall still, in this new triangle, have $m' = p'\psi : (A, B)$, and $n' = p'\psi : (A, B)$. Hence $m : m' :: n : n' :: p : p'$. Hence *in equiangular triangles, the sides opposite the equal angles are proportional.*

From this general proposition, we can deduce, as a particular case, the property assumed in the text for demonstrating Prop. 20. I. For the triangles AFG, AML have each two angles respectively equal, namely, the angle A common, and a right angle: hence they are equiangular, hence we have the proportion AF : AL :: AG : Am, by means of which Prop. 20 is completely proved.



The proposition concerning the square of the hypotenuse, we already know, is a consequence of that concerning equiangular triangles. Here then are three fundamental propositions of geometry,—that concerning the three angles of a triangle, that concerning equiangular triangles, and that concerning the square of the hypotenuse, which may be very simply and directly deduced from the consideration of functions. In the same way, the propositions relating to similar figures and similar solids may be demonstrated with great ease.



Let ABCD be any polygon. Having taken any side AB, upon AB as a base, form as many triangles ABC, ABD, &c. as there are angles C, D, E, &c. lying out of it. Put the base AB $= p$; let A and B represent the two angles of the triangle ABC, which are adjacent to the side AB; A′ and B′ the two angles of the triangle ABD, which are adjacent to the same side AB, and so on. The figure ABCDE will be entirely determined, if the side p with the angles A, B, A′, B′, A″, B″, &c. are known, and the number of data will in all amount to $2n - 3$, n being the number of the polygon's sides. This being granted, any side or line x, anyhow drawn in the polygon, and from the data alone which serve to determine this polygon, will be a function of those given quantities; and since $\frac{x}{p}$ must be a number, we may suppose $\frac{x}{p} = \psi : (\text{A, B, A}', \text{B}', \&\text{c.})$ or $x = p\,\psi\,(\text{A, B, A}', \text{B}', \&\text{c.})$, and the function ψ will not contain p. If with the same angles, and another side p', a second polygon be formed, the line x' corresponding or homologous to x will have for its value $x' = p'\,\psi : (\text{A, B, A}', \text{B}', \&\text{c.})$; hence $x : x' :: p : p'$. Figures thus constructed might be defined as *similar figures*; hence *in similar figures the homologous lines are proportional.* Thus, not only the homologous sides and the homologous diagonals, but also any lines terminating the same way in the two figures, are to each other as any other two homologous lines whatever.

Let us name the surface of the first polygon S; that surface is homogeneous with the square p^2; hence $\frac{\text{S}}{p^2}$ must be a number, containing nothing but the angles A, B, A′ B′ &c.; so that we shall have $\text{S} = p^2\varphi : (\text{A, B, A}', \text{B}', \&\text{c.})$; for the same reason, S′ being the surface of the second polygon, we shall have $\text{S}' = p'^2\varphi : (\text{A, B, A}'\ \text{B}', \&\text{c.})$ Hence $\text{S} : \text{S}' :: p^2 : p'^2$; hence *the surfaces of similar figures are to each other as the squares of their homologous sides.*

Let us now proceed to polyedrons. We may take it for granted, that a face is determined by means of a given side p, and of the several given angles A, B, C, &c. Next, the vertices of the solid angles which lie out of this face, will be determined each by means of three given quantities, which may be regarded as so many angles; so that the whole determination of the polyedron depends on one side, p, and several angles A, B, C, &c. the number of which varies according to the nature of the polyedron. This being granted, a line which joins to no vertices, or more generally, any line x drawn in a determinate manner in the polyedron, and from the data alone which serve to construct it, will be a function of the given quantities p, A, B, C, &c.; and since $\frac{x}{p}$ must be a number, the function equal to $\frac{x}{p}$, will contain nothing but the angles A, B, C, &c., and we may put $x = p\,\varphi : (\text{A, B, C, \&c.})$

The surface of the solid is homogeneous to p^2; hence that surface may be represented by $p^2\,\psi : (A, B, C, \&c.)$: its solidity is homogeneous with p^3, and may be represented by $p^3\,\Pi : (A, B, C, \&c.)$, the functions designated by ψ and Π being independent of p.

Suppose a second solid to be formed with the same angle A, B, C, &c., and a side p' different from p; and that the solids so formed are called *similar solids.* The line which in the former solid was $p\,\phi : (A, B, C, \&c.)$, or simply $p\,\phi$ will in this new solid become $p'\,\phi$; the surface which was $p^2\,\psi$ in the one, will now become $p'^2\,\psi$ in the other; and, lastly, the solidity which was $p^3\,\Pi$ in the one, will now become $p'^3\,\Pi$ in the other. Hence, first, *in similar solids, the homologous lines are proportional;* secondly, *their surfaces are as the squares of the homologous sides;* thirdly, *their solidities are as the cubes of those same sides.*

The same principles are easily applicable to the circle. Let c be the circumference, and s the surface of the circle whose radius is r; since there cannot be two unequal circles with the same radius, the quantities $\frac{c}{r}$ and $\frac{s}{r^2}$ must be determinate functions of r: but as these quantities are numbers, the expression of them cannot contain r; and thus we shall have $\frac{c}{r} = \alpha$, and $\frac{s}{r^2} = \beta$, α and β being constant numbers. Let c' be the circumference, and s' the surface of another circle whose radius is r'; we shall, as before, have $\frac{c'}{r'} = \alpha$, and $\frac{s'}{r'^2} = \beta$. Hence $c : c' :: r : r'$, and $s : s' :: r^2 : r'^2$; hence *the circumferences of circles are to each other as their radii, and their surfaces as the squares of those radii.*

Let us now examine a sector whose radius is r. A being the angle at the centre, let x be the arc which terminates the sector, and y the surface of that sector. Since the sector is entirely determined when r and A are known, x and y must be determinate functions of r and A; hence $\frac{x}{r}$, and $\frac{y}{r^2}$ are also similar functions. But $\frac{x}{r}$ is a number, as well as $\frac{y}{r^2}$; hence those quantities cannot contain r, and are simply functions of A; so that we have $\frac{x}{r} = \phi : A$, and $\frac{y}{r^2} = \psi : A$. Let x' and y' be the arc, and the surface of another sector, whose angle is A, and radius r'; we shall call those two sectors *similar*: and since the angle A is the same in both, we shall have $\frac{x'}{r'} = \phi : A$, and $\frac{y'}{r'^2} = \psi : A$. Hence $x : x' :: r : r'$, and $y : y' :: r^2 : r'^2$; hence *similar arcs,* or *the arcs of similar sectors are to each other as their radii; and the sectors themselves are as the squares of the radii.*

By the same method, we could evidently shew, that spheres are as the cubes of their radii.

In all this, we have supposed that surfaces are measured by the product of two lines, and solids by the product of three; a truth which it is easy to demonstrate by Analysis, in like manner. Let us examine a rectangle, whose sides are p and q; its surface, which must be a function of p and q, we shall represent by $\varphi : (p, q)$. If we examine another rectangle, whose dimensions are $p + p'$ and q, this rectangle is evidently composed of two others, of one having p and q for its dimensions, of another having p' and q; so that we may put $\varphi : (p + p', q) = \varphi : (p, q) + \varphi : (p', q)$. Let $p' = p$; we shall have $\varphi\ (2\ p, q) = 2\ \varphi c\ (p, q.)$ Let $p' = 2\ p$; we shall have $\varphi\ (3\ p, q) = \varphi\ (p, q) + \varphi\ (2\ p, q) = 3\ \varphi\ (p, q.)$ Let $p' = 3\ p$; we shall have $\varphi\ (4\ p, q) = \varphi\ (p, q) + \varphi\ (3\ p, q) = 4\ \varphi\ (p, q)$. Hence, generally, if k is any whole number, we shall have $\varphi\ (kp, q) = k\ \varphi\ (p, q)$, or $\frac{\varphi\ (p, q)}{p} = \frac{\varphi\ (kp, q)}{kp}$; from which it follows that $\frac{\varphi\ (p, q)}{p}$ is such a function of p as not to be changed by substituting in place of p any multiple of it kp. Hence this function is independent of p, and cannot include any thing except q. But for the same reason $\frac{\varphi\ (p, q)}{q}$ must be independent of q; hence $\frac{\varphi\ (p, q)}{pq}$ includes neither p nor q, and must therefore, be limited to a constant quantity α. Hence we shall have $\varphi\ (p, q) = \alpha\ p\ q$; and as there is nothing to prevent us from taking $\alpha = 1$, we shall have $\varphi\ (p, q) = p\ q$; thus the surface of a rectangle is equal to the product of its two dimensions.

In the very same manner, we could shew that the solidity of a right-angled parallelepipedon, whose dimensions are p, q, r, is equal to the product $p\ q\ r$ of its three dimensions.

We may observe, in conclusion, that the doctrine of functions, which thus affords a very simple demonstration of the fundamental propositions of geometry, has already been employed with success in demonstrating the fundamental principles of Mechanics. See *the Memoirs of Turin, vol. II.*

ADDITION TO NOTE II.

(Furnished by M. Legendre *for this Edition.)*

The celebrity which Professor Leslie of Edinburgh so justly enjoys, forbids me to pass over in silence the objections which this learned geometer has adduced against the foregoing theory, and particularly against my proof of the equation $C = \varphi\ (A, B)$, from which our theorem concerning the three angles of a triangle is derived.

The objections alluded to, first made their appearance in the second edition of Mr Leslie's *Elements of Geometry*, pp. 403 *et seq.* and though they were refuted, quite completely, as I think, in the equally severe and judicious criticism of that work, published by Mr Playfair, in the *Edinburgh Review*, vol. xx; though I replied to them in a private letter addressed to the author; yet Mr Leslie in his 3d edition, 1817, (*pp.* 292 *et seq.*), has again brought forward his objections, inserting along with them an extract from my letter * (which, as may be gathered from the end of the quotation, was not in any way designed for publication), and subjoining, in favour of his opinion, the testimony of a mathematician whom he does not name, but declares to be at the head of British geometers.

Without entering into any profound discussion of this question, I shall restrict myself to place before the reader the principal point of the difficulty. I am required to shew, in opposition to Mr Leslie's opinion, that a line, which is an absolute length, cannot be determined solely from angles, which are represented in calculation by their ratios to the right angle assumed as unity, that is to say, by numbers always included between 0 and 2. Thus the side c of a triangle cannot be determined solely from the angles A, B, C of this triangle; for these angles being only numbers, they can of themselves serve only for determining numbers. Accordingly, what information would you gain, if the value of the side c, as determined by calculating the function which represented it, were to come out $\frac{7}{10}$, for example? Is it $\frac{7}{10}$ of an inch, $\frac{7}{10}$ of a foot, $\frac{7}{10}$ of the earth's radius, or $\frac{7}{10}$ of the sun's distance? You cannot say, unless the question offer some other linear *datum* as unit, or which may serve for unit. With angles, however, the case is different, because the right angle is their natural unit, and any angle is completely determined, whenever we have discovered the value of its numerical ratio to the right angle.

If it is absurd to suppose, that the line c can be determined by the numerical quantities A, B, C alone; hence there can exist no equation between the quantities c, A, B, C. Hence the equation $C = \varphi(A, B, c)$, which is given immediately by the principle of superposition, can exist only on condition that c disappear from the second member of it; otherwise, we are taught by Analysis that an equation containing at once the four quantities A, B, C, c, would allow us to determine c by means of angles A, B, C. Hence we have simply $C = \varphi(A, B)$; that is, in every triangle, two given angles determine the third. From this property, it is easy to deduce the theorem concerning the three angles of a triangle, either by the demonstration given above, or by that which Mr Leslie himself proposes, which is very simple, but subject to exception in the case of the equilateral triangle.

Mr Leslie endeavours to draw an argument against this theory, from the case where the angle C, and the two sides a and b which contain it are given. Here, the third side c must be entirely determined by those data, which is expressed thus: $C = \varphi(a, b, C)$. Mr Leslie adds, (page 93, 3d edit.)

"But the angle C being heterogeneous to the sides a and b, cannot coalesce with them into an equation, and consequently the base c is simply a function of a and b, or it is the necessary result merely of the other two sides. Such is the extreme absurdity to which this sort of reasoning would lead."

But the only absurdity here is the reasoning employed by the objector, of which certainly I did not give him the example. Though an angle C, measured by a number, is undoubtedly heterogeneous to each of the quantities a and b which are lines, it is not on that account heterogeneous to their ratio $\frac{b}{a}$; and consequently there is no reason to expunge C from the function $\varphi\,(a, b, C.)$ In the present case, where the function is to represent the line $\varphi\, c$, this line c must be homogeneous with the lines a and b, and of one dimension; or what amounts to the same, it must be the product of a by a function of $\frac{b}{a}$ and of C; hence we shall have simply $c = a\,\psi\left(\frac{b}{a}, C\right)$; so that the function of three quantities is, in this case, reduced to a function of two quantities only. Such is the common doctrine of analytical writers: See the *Introduct. in Anàl.* of Euler, p. 65. It is evident, moreover, that the equation $c = a\,\psi\left(\frac{b}{a}, C\right)$ agrees completely with the trigonometrical formula, $c^2 = a^2\left(1 + \frac{b^2}{a} - \frac{2b}{a}\cos C\right)$. From this, therefore, no objection against my theory can be drawn, but rather a full and entire confirmation of it.

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* Two errors of the press have found their way into this extract; they may be corrected by putting c in place of C, page 296, line 31, &c.

To shew in a manner, if possible still clearer, that the law of homogeneity, combined with the known principles of the theory of functions, can lead to no results but what are accurate, let us consider an isosceles triangle formed by two equal sides a, a, and the angle C contained by them. Since this triangle is entirely determined by the given quantities a and C, the angle A opposite the side a must be a determinate function of the quantities a and C; we shall express it thus:— $A = \varphi\,(a, C.)$ Now, if the quantity a does not disappear from the function φ, then from the equation $A = \varphi\,(a, C)$, the value of a might be deduced in terms of A and C. But a line a, not being referred to any unit, cannot be equal to a function of two numbers A and C, which must itself be a number. Hence we have simply $A = \varphi\,(C)$; that is to say, in every isosceles triangle, the angle at the base is determined by the angle at the vertex, and conversely. From the vertex to the middle of the base, draw a straight line; it will divide the isosceles triangle into two equal right-angled triangles, and you will infer that in these right-angled triangles, one acute angle determines the other. Hence we conclude, as in the first demonstration, that the two acute angles of a right-angled triangle are together equal to a right angle; and then that in every triangle the sum of all the angles is equal to two right angles.

Let us now proceed to the chief objection made by the anonymous mathematician whose suffrage Mr Leslie brings to bear against me. He maintains that I have done nothing but elude the difficulty; and that my method, or what he calls the *mise en équation* of the problem, involves a supposition equivalent to Euclid's postulate, because I have considered as existing, and already constructed a triangle in which the angles amount to a sum as near two right angles as we please. If I had indeed made this supposition, my critic were undoubtedly in the right; but in reality, I make no supposition. I reason with regard to any triangle already constructed, and actually existing; I study its properties, and find from studying them, that in every triangle the sum of all the angles is equal to two right angles. So soon as this principle is established, I can easily (as I have done in the third edition of my Elements, Prop. 24. I.), construct a triangle having a given base *c*, and two adjacent angles A and B, whose sum differs from two right angles as little as we please; and by this construction, Euclid's postulate is demonstrated in the most rigorous manner.

I shall extend my observations no farther; I might even have forborne to write on this subject at all, since M. Maurice of the Academy of Sciences has undertaken the task of replying to the objections of Mr Leslie and his learned correspondent, in the *Bibliothèque Universelle* of Geneva, Oct. 1819; and his dissertation (excepting one or two passages, perhaps liable to dispute, but having no influence on the question) completely fulfils the object of its author, and establishes in a manner as solid as luminous, the theory which Mr Leslie has attempted to overturn.

May I observe, in conclusion, that Mr Leslie, who has hitherto appeared in the character of an assailant, has not sufficiently secured his own defence, having left without reply the very strong objection alleged against his demonstration of Prop. 22, by Mr Playfair, at p. 88 in the volume of the *Edinburgh Review* already quoted. In reality, this demonstration supposes, that through a given point, no more than one parallel to a given line can be made to pass, or that there is only one position in which the line meant to be drawn will not meet the given line. Such an assumption is identically the same as Euclid's postulate. There is only this difference between Mr Leslie's method and that of Euclid, that the ancient geometer does not dissemble the difficulty, but presents it, on the contrary, in all its breadth, and requires to have that granted which he cannot prove; while the modern geometer envelopes the difficulty in a shadow of demonstration which, though doubtless it has seduced himself, is certainly any thing but rigorous. It is difficult to conceive how such a mistake could proceed from a mathematician so well versed as Mr Leslie is in the geometry of the ancients; who has shewn himself acquainted with all its most subtle refinements, and has himself invented many demonstrations which the ancients would not have been ashamed to own. One would have expected him to look more narrowly into a subject, which has formed the great difficulty of geometers ancient as well as modern; and not to give out as rigorous a demonstration, which is very far from being so.

⁂ We subjoin the paper of M. le Baron Maurice, alluded to above, which M. Legendre has taken the trouble to alter in a few points, and adapt for appearing in this Translation.—Ed.

On the Application of the Algorithm of Functions to demonstrate the fundamental Propositions of Geometry; in Reply to some Criticisms.—Extracted from the Bibl. Univ. *Geneva, October* 1819.

In a justly valued work (*Elements of Geometry, by John Leslie,* 3d edit. 8vo, Edinburgh, 1817), a great number of objections, more or less specious, are combined together, and directed against the force of the demonstration, which M. Legendre, in Note II. of his Geometry, gives of the fundamental proposition, that, *in every rectilineal triangle, the sum of the three angles is equal to two right angles.* This demonstration is the more remarkable, as it makes no use of the theory of parallels, but, on the other hand, itself serves rigidly to establish that theory afterwards; and as, being founded solely on the principles of *superposition* and of *homogeneity,* the most palpable and evident of all principles, it shares to the full extent in their correctness.

Accordingly, on examining with attention the objections brought forward by Mr Leslie, and by one of his mathematical correspondents, whom he does not name, but speaks of as being at the head of British mathematicians (p. 294 *of his work*), we shall not fail to discover that their objections, if well-founded, would overthrow the two principles we have mentioned as well as the demonstration which is derived from them. In this paper we mean to establish M. Legendre's demonstration upon an indisputable basis, and to point out, as we proceed, an immediate consequence from it.

Let us first recollect what it is that forms the ground-work of this demonstration: we shall then present the objections and our replies to them.

According to M. Legendre, since the principle of superposition shews that a triangle is determined when one of its sides and the angles adjacent to this side are known, it follows that the third angle must be entirely determined by these three elements which are supposed to be known, or using the terms of Analysis, that it must be a determinate, though as yet an unknown *function* of them. Thus naming the angles of this triangle A, B, C, and the sides opposite to them a, b, c, we have the symbolical equation $C=\varphi(A, B, c)$. Now A, B, C being determinate angles, are also in a determinate ratio with the total angular quantity, or with the right angle, which is a fourth part of it; they are consequently real numbers, if we assume, as the angular *unit,* the right angle for example. Hence, if the line c could remain in the preceding equation, we might deduce from it $c=f(A, B, C)$, or that a line is a function of the angular unit; an absurd consequence incompatible with the law of homogeneity. By virtue of this law, therefore, we must admit that the proposed equation will reduce itself to this: $C=\varphi(A, B)$, which signifies that the third angle of a triangle is entirely determined by means of the other two; therefore, two angles of one triangle being equal to two of another, the third in the one must be equal to the third in the other; from which the author very easily infers (by a mode which has never been attacked) that the sum of the three angles in a rectilineal triangle is always equal to two right angles. The function $\varphi(A, B)$ is therefore an algebraical function equal to $2-A-B$; and the proposed function becomes $C+B+A=2$, a rigorous equation among abstract numbers.

If the triangle were spherical, this reasoning would not be applicable. In spherical triangles there is another element, the radius of the sphere, to which all the sides must be compared. The proposed equation in that case would be $C=\varphi\left(A, B, \frac{c}{r}\right)$; and there is no absurdity in supposing $\frac{c}{r}$, an *abstract* number, to be a function of angles, since this function itself is an abstract number.

Such is very nearly M. Legendre's demonstration: let us now see Mr Leslie's objections.

According to Mr Leslie this demonstration will not bear a rigid examination. "Many quantities, in fact, appear to result from the combined relation of other quantities that are altogether heterogeneous. Thus the space, which a moving body describes, depends on the joint elements of time and velocity, things entirely distinct in their nature; and thus the length of an arc of a circle is compounded of the radius, and of the angle it subtends at the centre, which are obviously heterogeneous magnitudes. For aught we previously knew to the contrary, the base c might, by its combination with the angles A and B, modify their relation, and thence affect the value of the vertical angle C. In another parallel case the force of this remark is easily perceived. Thus when the sides a, b and their contained angle C are given, the triangle is determined, as the simplest observation shews. Wherefore the base c is derived solely from these data, or $c = \varphi : (a, b, C)$. But the angle C being heterogeneous to the sides, a and b cannot coalesce with them into an equation, and consequently the base c is simply a function of a and b, or it is the necessary result merely of the other two sides.

"In other words, as the third angle of a triangle depends on the other two angles, so the base of a triangle must have its magnitude determined by the lengths of the two incumbent sides. Such is the extreme absurdity to which this sort of reasoning would lead! In both these instances, indeed, the conclusion is admitted by implication, only in the one it is consistent with truth, while in the other it is palpably false. * * * The whole stress of the argument, it may be perceived, lies in the distinction which M. Legendre endeavours to establish between angles and lines,—a distinction which I hold at bottom to be merely arbitrary. Angles and lines are both equally real quantities, though of different kinds; they are capable of being measured, and consequently represented by numbers, by referring each of them to some definite measure or unit of its own denomination. Angles are measured or expressed numerically by angles, and lines by lines. It is true that the mensuration of angles is facilitated by a reference to a subdivision of the circuit or entire revolution; yet even this mode of denoting angular magnitude is evidently only conventional. As standards for measuring straight lines, nature has furnished the limbs of the human body, and the extent of our globe itself. Such units of mensuration are not indeed very definite or readily attainable; but they are not therefore the less real or prominent. Nor is there any essential difference in principle between the expressing of an angle by degrees, of which 360 or 400 are contained in a complete revolution, and the denoting of a straight line on the French system, for instance, by the number of *metres* it includes, each of which is the forty millionth part of the entire circumference of the earth. Angles and lines hence present to the mind no radical or absolute discrimination, and therefore the argument grounded on such a distinction must lose all its efficacy."—(*Leslie's Geometry*, pp. 293, 294, 297.)

Such are *all* the criticisms which belong to the ingenious geometer and natural philosopher of Edinburgh. He adds (p. 294), that, with a single exception, he is acquainted with no geometer of any eminence in Britain, who does not admit the fallacy of the argument employed in the demonstration before us. Nevertheless, it is easy, in reading his objections, to perceive that they rest on a palpable error; he confounds, throughout, the natural homogeneity of quantities, with the principle of homogeneity observed in every relation, founded on the laws of the calculus; which two principles are widely different.

Let us begin by resuming this latter principle, and establishing it firmly.

A very little reflection will convince us that no equation can subsist except between homogeneous quantities, that so it may afterwards be reduced to a relation between abstract numbers, when each term of this equation shall have been delivered (if necessary), by division from the *unit peculiar* to the nature of the quantities considered. Thus no question of arithmetic or algebra ever occurred, in which the unknown quantities were aught but abstract numbers; and it is in knowing how to *write* the questions, so as to obtain this end, that the art of the analyst consists. So, likewise, in questions of Mechanics, no times or velocities, for example, are ever actually assigned for the value of certain spaces: in such questions, where it seems at first that heterogeneous quantities are combined together, it may still be discovered, that by the skill of the analyst, the true elements of the calculation at length become simply the ratios of these different quantities to their respective units, and the ratios of these ratios; so that, even in questions of this sort, no actual relation is ever definitively established, except between abstract numbers.

Consequently, in the case before us, when we have arrived at the general relation expressed by the symbolical equation $C=\varphi\ (A,.B,\ c,)$ it is *rigorously essential* to its existence that it be capable of being reduced to a relation among abstract numbers. Now, if only the angles A, B, C entered into it, there would be no difficulty: for since each of them expresses a multiple or submultiple of the angular unit, this unit may be made to disappear by means of division. But if the straight line *c* enters also, this relation becomes manifestly absurd, since there occur in it two heterogeneous units, which cannot both be made to disappear from the calculation. If, however, another straight line as *a* or *b*, other sides of the triangle, or if *a* and *b* occurred at once in the equation, this impossibility of the equation would no longer exist; since we are allowed to suppose that only the ratios of those lines enter into the function; and as these ratios are abstract numbers, the equation would rigorously subsist, inasmuch as these ratios would be factors of the angles or of certain functions of the angles, because in that case, the angular unity disappearing from the calculation, there would only remain a relation between abstract numbers. Hence it is because the line *c* is *the only* straight line which occurs in the proposed relation, that we are rigorously authorised *a priori* to eliminate this line from it, as a quantity which cannot remain without leading to a manifest absurdity.

We can also give an irrefragable mathematical proof, *a posteriori*, that this elimination must take place, in the case we are now considering. Suppose the triangle in question were a spherical one; we have seen that the symbolical equation becomes $C=\varphi\left(A, B, \frac{c}{r}\right)$, and that we cannot here reason as we did in the case of the rectilineal triangle, $\frac{c}{r}$ being a real number. But if r becomes infinite, that is to say, if the sphere becomes a plane, and so our *spherical* triangle is changed into a *rectilineal* one, then it is evident that C, A, B, the determined angles being *finite* multiples or submultiples of the angular unit, which itself bears a finite proportion to the total angular quantity, those angles can no longer be compared with the quantity $\frac{c}{\infty}$, and this, whatever be the magnitude of c; whence it is evident that the magnitude of the angle C cannot be affected by that of the side c, and that hence the proposed relation must of necessity reduce itself to the rigorous equation $C=\varphi$ (A, B.)

It is objected, that on the same principle, from this other symbolical relation $c=\varphi(a, b, C)$, it will also be necessary to eliminate C, and therefore to arrive at an absurd conclusion; an objection which proves still more clearly, that Mr Leslie has confounded the homogeneity of quantities with the principle of homogeneity in the calculus. But M. Legendre has shewn sufficiently, that such an elimination is not in this case necessary; and for proof he exhibits the trigonometrical relation

$$\cos C=\frac{1+\frac{b^2}{a^2}-\frac{c^2}{a^2}}{2\frac{b}{a}},$$ an equation between abstract numbers. Now this example is not, as might in the first glance be supposed, at variance with the principles here laid down: if the angular unit, though not affecting all the terms, has disappeared, the reason is, that the proposed relation turns out to establish, that a certain function of the sides of a triangle is a function of one of its angles; and as this latter function may be transcendental, and consequently, equal to the cosine of the arc which corresponds to that angle, it follows, that just as we have the general relation $Angle=\frac{Arc}{Radius}$, so also we have a certain cosine (and every cosine must be an abstract number) equal to a certain algebraical function of the ratios between the sides. By writing the trigonometrical equation under the form $\frac{Arc\ C}{r}=\frac{Arc}{r}\left\{\cos=\frac{1+\frac{b^2}{a^2}-\frac{c^2}{a^2}}{2\frac{b}{a}}\right\}$, *

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* This supposes $r=1$, and the angle C to be measured by the arc C taken in a circle whose radius is 1. *Note by M. Legendre.*

we should see, in like manner, how in this case the proposed equation may be reduced to a relation between abstract numbers.

What has now been said is enough, without doubt, completely to illustrate the question. We may add, however, that if we chose to ascend to a more elevated analytical principle, it were easy to shew, that between angles and *one* straight line, there cannot exist any relation from which the latter, as a function of the former, can be determined. These quantities considered as *magnitudes* destined to enter into our calculations, are not homogeneous, when referred to the wholes of which they respectively form part. The angle is a portion of a finite whole, the straight line a portion of an infinite whole; so that *every given angle is a finite quantity,* whilst *every given straight line is a quantity infinitely small,* and *only the ratios of given straight lines can enter into our calculations with given angles.**

It is upon this principle only that we can explain the *analytical* paradox which seems to result from the relation $C = \varphi(A, B, c)$ not being capable of admitting the solution $c = \varphi(A, B, C)$. By the foregoing considerations, it is evident that this solution is impossible, and the geometrical indetermination of the question corresponds to this analytical impossibility.

The objections stated above may now be considered as entirely dissipated; a few reflections may, however, be subjoined.

Mr Legendre is reminded, that heterogeneous quantities are considered simultaneously in the science of Mechanics. But every one knows, that in this science, all equations expressing any relation between space and time, will, besides these two elements, of necessity include *a line-unit and a time-unit.* The equation considered, therefore, will not be simply between s, s' - - and t, t'; but in truth, between $\frac{s}{S}$, $\frac{s'}{S'}$ - - - and $\frac{t}{T}$, $\frac{t'}{T'}$, - - -, S and T being the two units of space and time; so that no equation of this kind will ever exist, except between abstract numbers. And if velocities also are supposed to enter into this equation, is it not, in like manner, well known, that these velocities, mere conceptions of our own mind, are nothing but abstract numbers resulting from the ratios of abstract numbers, as $\frac{s}{S}$ and $\frac{t}{T}$?

Another objection is sought for in the dependence which exists between an arc of a circle, its radius and the angle to which it correthe question, he meets, and quite gratuitously, with all the difficulties inherent in the *direct* establishment of the theory of parallel lines, and arising from the idea of infinity, which is immediately connected with that of the straight line in general; whilst M. Legendre, naturally setting out from the *existing* triangle (and we have seen that this proceeding cannot be objected to without completely changing the state of the question), finds means to deduce this very theory from the proposition concerning the sum of the three angles. He thus most skilfully eludes the difficulty, by taking advantage of the indisputable equation $C=\varphi(A, B, c)$ in order to get rid of every idea of infinity by means of eliminating the straight line c; an operation depending, as we have seen, on a principle as rigorously true as any of those which have been hitherto admitted in geometry.

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* This is a very subtle and very just metaphysical idea: it is, at the same time, strictly analytical; and explains why c stood alone in the equation $C = \varphi(A, B, c)$, where c may be suppressed, because it is infinitely little, or nothing. It also explains why, in the case where two lines a and b are found in an equation containing no other quantities but numbers; these two lines cannot exist in the equation except by their ratio $\frac{b}{a}$, so that making $b = ap$, the quantity, a would of necessity disappear from the equation. *Note by M. Legendre.*

The force of M. Legendre's demonstration is therefore derived from the fact, that a necessary law of the calculus, when applied to any relation indicated by geometry, is to retain in this relation nothing but quantities of a comparable magnitude; and the art of its author, on this occasion, has been to put under the decision of the calculus which implied the necessity of such an elimination, the solution of a question which geometrical considerations *alone* left in its primitive complexity.

In fine, if we make the superfluous remark, that the same mode of reasoning, when employed to demonstrate several other very important propositions, in the remaining part of M. Legendre's second Note, has always guided its author to correct results, whilst no one has succeeded in legitimately deducing from it a single consequence in itself incorrect, we shall be warranted in concluding from all that precedes, that *the demonstration attacked in Mr Leslie's work is perfectly rigorous.* The algorithm of functions had previously been employed with success in establishing the fundamental principles of Mechanics (see the *Miscell. Taurin.* tom. I. and II.); but this new application of it does not yield to those which have gone before; indeed, such a result was to be expected from so happily introducing into the fundamental relation which geometry furnishes, a principle so evident and incontestable as that of *homogeneity.*

F. M

NOTE III.

On the Approximation employed in Prop. 16. IV.

So soon as we have found a radius too great and a radius too little which agree in their first ciphers, the calculation may be completed in a very speedy manner by means of an algebraical formula.

Let a be the defective radius, and b the excessive one, their difference being small; let a' and b' be the radii next in order deduced by the formulas $b'=\sqrt{ab}$, $a'=\sqrt{\left(a\,.\,\frac{a+b}{2}\right)}$. What we are in quest of is the last term of the series a, a', a'', &c., which at the same time will be the last of the series b, b', b'', &c. Let this last term be named x, and put $b=a\,(1+\omega)$; we can suppose $x=a\,(1+\mathrm{P}\,\omega+\mathrm{Q}\omega^2+\&c.)$, P and Q being indeterminate coefficients. Now the values of b' and a' give

$$b'=a\,(1+\tfrac{1}{2}\omega-\tfrac{1}{8}\omega^2+\&c.)\,;$$
$$a'=a\,(1+\tfrac{1}{4}\omega^2-\tfrac{1}{32}\omega^2+\&c.)$$

And if, in like manner, we put $b'=a'\,(1+\omega')$, we shall have

$$\omega'=\tfrac{1}{4}\omega-\tfrac{5}{32}\omega^2\ \&c.$$

But the value of x must be the same whether the series a, a', c'', &c. begins with a or a'; hence we shall have

$$a\,(1+\mathrm{P}\omega+\mathrm{Q}\omega^2+\&c.)=a'\,(1+\mathrm{P}\omega'+\mathrm{Q}\omega'^2+\&c.)$$

Substitute, in this equation, the values of a' and ω' in terms of a and ω; and comparing the similar terms, we shall deduce from it $\mathrm{P}=\frac{1}{3}$, and $\mathrm{Q}=-\frac{1}{15}$; hence

$$x=a\,(1+\tfrac{1}{3}\omega-\tfrac{1}{15}\omega^2).$$

If the radii a and b agree in the first half of their ciphers, the term ω^2 may be rejected in the calculation, and the preceding value will be reduced to $x=a\,(1+\frac{1}{3}\omega)=a+\dfrac{b-a}{3}$. Thus making $a=1.1282657$, and $b=1.1286063$, we shall immediately deduce from it $x=1.1283792$.

If the radii a and b agree only in the first third of their ciphers, we shall have to take in the three terms of the preceding formula; and thus making $a=1.1265639$, and $b=1.1320149$, we shall find $x=1.1283791$.

We might suppose a and b to be still farther different; but in that case it would be requisite to calculate the value of x with a greater number of terms.

The approximation exhibited in Prop. 14, the author of which is James Gregory, is susceptible of similar abridgments. We refer to Gregory's book, entitled *Vera Circuli et Hyperbolæ Quadratura*, a work of great merit, considering the time when it appeared.

NOTE IV.

Shewing that the Ratio of the Circumference to the Diameter, and also its square, are irrational Numbers.

Let us examine the infinite series

$$1+\frac{a}{z}+\frac{1}{2}\cdot\frac{a^2}{z.z+1}+\frac{1}{2.3}\cdot\frac{a^3}{z.z+1.z+2},\ \&c.$$

of which the general term is $\dfrac{1}{1.2.3\ldots n}\cdot\dfrac{a^n}{z.z+1.z+2\ldots.(z+n-1)}$

and suppose that $\varphi : z$ represents the sum of it. Putting $z+1$ in place of z, $\varphi\,(z+1)$ will in like manner be the sum of the series,

$$1+\frac{a}{z+1}+\frac{1}{2}\cdot\frac{a^2}{z+1\,.\,z+2}+\frac{1}{2.3}\cdot\frac{a^3}{z+1.z+2.z+3}+\&c.$$

Subtract the one of these series from the other, term by term; we shall have $\varphi : z - \varphi : (z+1)$ for the sum of the remainder, which, in its expanded form, will be

$$\frac{a}{z.z+1}+\frac{a^2}{z\,.\,z+1\,.\,z+2}+\frac{1}{2}\cdot\frac{a^3}{z.z+1.z+2.z+3}+\&c.$$

But this remainder may be put under the form

$$\frac{a}{z.z+1}\cdot(+\frac{a}{z+2}+\frac{1}{2}\cdot\frac{a^2}{z+2.z+3}+\&c.);$$

and then it is reducible to $\frac{a}{z.z+1}\,\varphi : (z+2)$. Hence, generally, we shall have

$$\varphi : z - \varphi : (z+1) = \frac{a}{z.z+1}\,\varphi : (z+2).$$

Divide this equation by $\varphi : (z+1)$, and to simplify the result, let $\psi : z$ be a new function of z, such that $\psi : z = \frac{a}{z}\cdot\frac{\varphi : (z+1)}{\varphi : (z)}$; we may then put $\frac{a}{z\,\psi : z}$ instead of $\frac{\varphi : z}{\varphi : (z+1)}$, and $\frac{(z+1)\,\psi : (z+1)}{a}$ instead of $\frac{\varphi : (z+2)}{\varphi : (z+1)}$. This substitution being made, we shall have

$$\psi : z = \frac{a}{z+\psi : (z+1)},$$

But by successively inserting $z+1$, $z+2$, &c., in place of z in this equation, there will result from it

$$\psi : (z+1) = \frac{a}{z+1+\psi : (z+2)},$$

$$\psi : (z+2) = \frac{a}{z+2+\psi : (z+3)}; \&c.$$

Hence the value of $\psi : z$, may be expressed by the continued fraction,

$$\varphi : z = \cfrac{a}{z+\cfrac{a}{z+1+\cfrac{a}{z+2+\&c.}}}$$

Reciprocally this continued fraction, produced to infinity, has for its sum $\psi : z$, or its equal $\frac{a}{z}\cdot\frac{\varphi : (z+1)}{\varphi : z}$; which sum, developed into its two ordinary series, becomes

$$\frac{a}{z}\cdot\frac{1+\frac{a}{z+1}+\frac{1}{2}\cdot\frac{a^2}{z+1\,.\,z+2}+\&c.}{1+\frac{a}{z}+\frac{1}{2}\cdot\frac{a^2}{z\,.\,z+1}+\&c.}$$

Now suppose $z=\frac{1}{2}$; the continued fraction will become

$$\cfrac{2a}{1+\cfrac{4a}{3+\cfrac{4a}{5+\&c.,}}}$$

in which all the numerators, except the first, are equal to $4a$, and the denominators form the series of odd numbers, 1, 3, 5, 7, &c. The value of this continued fraction may, therefore, be expressed by

$$2a\cdot\frac{1+\frac{4a}{2.3}+\frac{16a^2}{2.3.4.5}+\frac{64a^5}{2.3..7}+\&c.}{1+\frac{4a}{2}+\frac{16a^2}{2.3.4}+\frac{64a^5}{2.3...6}+\&c.}$$

But these series have a relation to some admitted formulas; and it is well known that, putting e for the number whose hyperbolic logarithm is 1, the foregoing expression becomes

$\frac{e^{2\sqrt{a}}-e^{-2\sqrt{a}}}{e^{2\sqrt{a}}+e^{-2\sqrt{a}}}\cdot\sqrt{a}$; so that we shall have, generally,

$$\frac{e^{2\sqrt{a}}-e^{-2\sqrt{a}}}{e^{2\sqrt{a}}+e^{-2\sqrt{a}}}\cdot 2\sqrt{a}=\cfrac{4a}{1+\cfrac{4a}{3+\cfrac{4a}{5+\&c.}}}$$

From this, two principal formulas are derived, according as a is positive or negative. First, let $4a=x^2$; we shall have

$$\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\cfrac{x}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\&c.}}}$$

Next let $4a=-x^2$; and agreeably to the demonstrated formula

$\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}=\sqrt{-1}\,.\,\text{tang.}\,x$, we shall have

$$\text{tang.}\,x=\cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{x^2}{7-\&c.}}}}$$

This formula will serve as the basis of our demonstration. Before proceeding to it, however, we must prove the two following Lemmas.

Lemma I. *Suppose*

$$\cfrac{m}{n + \cfrac{m'}{n' + \cfrac{m''}{n'' + \&c.}}}$$

to be a continued fraction prolonged to infinity, in which all the numbers m, n, m′, n′ *are positive or negative integers; if the component fractions* $\frac{m}{n}, \frac{m'}{n'}, \frac{m''}{n''}$, *&c. are all less than unity, then will the total value of the continued fraction be of necessity an irrational number.*

In the first place, this value is less than unity. For, without interfering with the general applicability of the continued fraction, we are at liberty to suppose all the denominators, n, n', n'', &c. to be positive; in which case, taking a single term of the proposed series, we shall, by hypothesis, have $\frac{m}{n} \angle 1$. Taking the first two, by reason of $\frac{m'}{n'} \angle 1$, it is evident that $n + \frac{m'}{n'}$, is greater than $n - 1$: but m is less than n; and since they are both integers, m will also be less than $n + \frac{m'}{n'}$. Hence the value which results from the two terms

$$\cfrac{m}{n + \cfrac{m'}{n'}}$$

is less than unity. Calculate three terms of the proposed continued fraction; and in the first place, as we have just seen, the value of the part

$$\cfrac{m'}{n' + \cfrac{m''}{n''}}$$

will be less than unity. Call this value ω; it is plain that this $\frac{m}{n + \omega}$ will still be less than unity: hence the value which results from the three terms

$$\cfrac{m}{n + \cfrac{m'}{n' + \cfrac{m''}{n''}}}$$

is less than unity. By continuing the same process, it will appear, that whatever number of terms in the proposed continued fraction be calculated, the value resulting from them is less than unity; hence the total value of the fraction prolonged to infinity, is also less than unity. It cannot be equal to unity except in the single case, when the proposed fraction had the form

$$\cfrac{m}{m+1-\cfrac{m'}{m'+1-\cfrac{m''}{m''+1-\&c.}}}$$

in every other case it is less.

This being proved, if the value of the continued fraction is not admitted to be an irrational number, suppose it to be a rational number, to be $\frac{B}{A}$, for example, B and A being any integers; we shall then have

$$\frac{B}{A}=\cfrac{m}{n+\cfrac{m'}{n'+\cfrac{m''}{n''+\&c.}}}$$

Let C, D, E, &c. be indeterminate quantities, such that

$$\frac{C}{B}=\cfrac{m'}{n'+\cfrac{m''}{n''+\cfrac{m'''}{n'''+\&c.}}}$$

$$\frac{D}{C}=\cfrac{m''}{n''+\cfrac{m'''}{n'''+\cfrac{m^{\text{IV}}}{n^{\text{IV}}+\&c.}}}$$

and so on to infinity. These different continued fractions having all their terms less than unity, their values or sums $\frac{B}{A}, \frac{C}{B}, \frac{D}{C}, \frac{E}{D}$, &c. will be less than unity, as we have proved above; and thus we shall have $B \angle A$, $C \angle B$, $D \angle C$, &c.; so that the series A, B, C, D, E, &c. goes on decreasing to infinity. But the combination of the continued fractions we are treating of gives

$$\frac{B}{A}=\frac{m}{n+\frac{C}{B}}; \text{ whence results } C = mA - nB,$$

$$\frac{C}{B}=\frac{m'}{n'+\frac{D}{C}}; \text{ whence results } D = m'B - n'C,$$

$$\frac{D}{C}=\frac{m}{n''+\frac{E}{D}}; \text{ whence results } E = m''C - n''D,$$

&c. &c.

And since the two first numbers A and B are integers by hypothesis, it follows that all the others C, D, E, &c. which were hitherto undetermined, are also integers. Now it implies a contradiction to suppose that an infinite series A, B, C, D, E, &c. cannot at once be decreasing and composed of integer numbers; for, besides, no one of the numbers A, B, C, D, E, &c. can be zero, since the proposed continued fraction extends to infinity, and therefore the sums represented by $\frac{B}{A}$, $\frac{C}{B}$, $\frac{D}{C}$, &c. must always be something. Hence our hypothesis, that the sum of the proposed continued fraction was equal to a rational quantity, cannot stand; hence that sum is of necessity a rational number.

Lemma II. *The same suppositions continuing as in the former Lemma, if the component fractions* $\frac{m}{n}$, $\frac{m'}{n'}$, $\frac{m''}{n''}$, &c. *are of any magnitude whatever at the beginning of the series, provided after a certain interval they become less than unit; we assert, that the proposed continued fraction, if it still extends to infinity, will have an irrational value.*

For if, reckoning from $\frac{m'''}{n'''}$ for example, all the fractions $\frac{m'''}{n'''}$, $\frac{m^{\text{IV}}}{n^{\text{IV}}}$, $\frac{m^{\text{V}}}{n^{\text{V}}}$, &c. to infinity, are less than unit, then by Lemma I, the continued fraction

$$\cfrac{m'''}{n''' + \cfrac{m^{\text{IV}}}{n^{\text{IV}} + \cfrac{m^{\text{V}}}{n^{\text{V}} + \&c.}}}$$

will have an irrational value. Call this value ω, the proposed continued fraction will become

$$\cfrac{m}{n + \cfrac{m'}{n' + \cfrac{m''}{n'' + \omega}}}$$

But if we successively put

$$\frac{m''}{n'' + \omega} = \omega', \quad \frac{m'}{n' + \omega'} = \omega'', \quad \frac{m}{n + \omega''} = \omega''',$$

it is evident that ω being irrational, all the quantities ω', ω'', ω''', must be so likewise. But ω''', the last of these, is equal to the proposed continued fraction, hence the value of this fraction is irrational.

We are now in a condition to resume our subject, and demonstrate this general proposition.

THEOREM.

If an arc is commensurable with the radius, its tangent will be incommensurable with that radius.

Put the radius $=1$, and the arc $x = \frac{m}{n}$, m and n being whole numbers; the formula found above, making the proper substitution, will give us,

$$\text{tang.}\ \frac{m}{n} = \cfrac{m}{n - \cfrac{m^2}{3n - \cfrac{m^2}{5n - \cfrac{m^2}{7n - \&c.}}}}$$

Now this continued fraction falls under Lemma II; for since the denominators $3n$, $5n$, $7n$, &c. increase continually, whilst the numerator m^2 continues of the same magnitude, the component fractions will evidently be, or at least will soon become, less than unit; hence the value of tang. $\frac{m}{n}$ is irrational; hence *if the arc is commensurable with the radius, its tangent will be incommensurable.*

From this, we deduce, as an immediate consequence, the proposition which forms the object of this Note. Let π be the semicircumference of which the radius is 1; if π were rational, the arc $\frac{\pi}{4}$ would be so too, and therefore its tangent would be irrational: but the tangent of the arc $\frac{\pi}{4}$ is well known to be equal to the radius 1; hence π cannot be irrational. Hence *the ratio of the circumference to the diameter is an irrational number.*[*]

It is probable that this number π is not even included among algebraical irrational quantities, in other words, that it cannot be the root of an algebraical equation having a finite number of terms with rational coefficients: but a rigorous demonstration of this seems very difficult to find; we can only shew that the square of π is also an irrational number.

Thus, if in the continued fraction, which denotes tang. x, we put $x=\pi$, since tang. $\pi=0$, we must have

$$0 = 3 - \cfrac{\pi^2}{5 - \cfrac{\pi^2}{7 - \cfrac{\pi^2}{9 - \&c.}}}$$

But if π^2 were rational, and we had $\pi^2=\frac{m}{n}$, m and n being whole numbers, there would result from it

$$3 = \cfrac{m}{5n - \cfrac{m}{7n - \cfrac{m}{9n - \cfrac{m}{11n - \&c.}}}}$$

Now, as this continued fraction evidently comes under Lemma II., its value also must be irrational, and cannot be equal to the number 3. Hence *the square of the ratio between the circumference and the diameter is an irrational number.*

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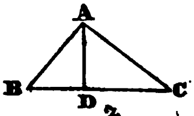
[*] This proposition was first demonstrated by Lambert, in the Memoirs of Berlin, *anno* 1761.

NOTE V.

Containing the analytical solution of various problems concerning the triangle, the inscribed quadrilateral, the parallelepipedon, and the triangular pyramid.

PROBLEM I.

Given the three sides of a triangle, to find its surface, with the radii of the inscribed and the circumscribed circle.



Put the sides BC$=a$, AC$=b$, AB$=c$; if from the vertex A, the perpendicular AD is let fall on BC the opposite side, then (Prop. 12. III.), we shall have AC^2 = AB^2 + BC^2 − 2BC × BD; hence BD$=\dfrac{a^2+c^2-b^2}{2a}$. This value gives AB^2−BD^2, or AD$^2=c^2-\left(\dfrac{a^2+c^2-b^2}{2a}\right)^2=\dfrac{4a^2c^2-(a^2+c^2-b^2)^2}{4a^2}$; hence AD$=\dfrac{\sqrt{[4a^2c^2-(a^2+c^2-b^2)^2]}}{2a}$. Let S be the area of the triangle, we shall have S$=\frac{1}{2}$ BC × AD; hence

$$S=\tfrac{1}{4}\sqrt{[4a^2c^2-(a^2+c^2-b^2)^2]}=\tfrac{1}{4}\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}.$$

This formula may farther be reduced to a form more convenient for logarithmic calculation. To effect this, we must consider that the quantity $4a^2c^2-(a^2+c^2-b^2)^2$ is the product of the two factors $2ac+(a^2+c^2-b^2)$ and $2ac-(a^2+c^2-b^2)$; the first $=(a+c)^2-b^2=(a+c+b)\,(a+c-b)$; the second $=b^2-(a-c)^2=(b+a-c)\,(b-a+c)$; hence we shall have

$S=\tfrac{1}{4}\sqrt{[(a+b+c)\,(a+b-c)\,(a+c-b)\,(b+c-a)]}$. Lastly, by making $\dfrac{a+b+c}{2}=p$, which gives $a+b+c=2p$, $a+b-c=2p-2c$, $a+c-b=2p-2b$, $b+c-a=2p-2a$, we shall have, still more simply,

$$S=\sqrt{(p\,.\,p-a\,.\,p-b\,.\,p-c)}.$$

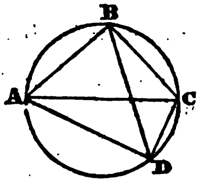
Thus it appears, that to obtain the surface of a triangle whose three sides are given, we must take the half sum of those three sides; from this half sum subtract each side separately, which will give us three remainders; multiply these three remainders and the half sum all together; and finally, extract the square root of the product: this root will be the area of the triangle.

Now, let z be the radius of the circle which circumscribes the triangle, and u the radius of the circle inscribed in it; by Prop. 23. III., we shall have

$z = \frac{\frac{1}{4}abc}{S}$ and $u = \frac{2S}{a+b+c} = \frac{S}{p}$; or, substituting the value found for S, we shall find $z = \frac{\frac{1}{4}abc}{\sqrt{(p.p-a.p-b.p-c)}}$, $u = \sqrt{\left(\frac{p-a.p-b.p-c}{p}\right)}$.

PROBLEM II.

Given the four sides of the inscribed quadrilateral, to find the radius of the circle, the surface of the quadrilateral, and its angles.



Put the given sides $AB=a$, $BC=b$, $CD=c$, $DA=d$, and the unknown diagonals $AC=x$, $BD=y$; by Prop. 33. III., we shall have $xy = ac+bd$ and $\frac{x}{y} = \frac{ad+bc}{ab+cd}$; whence we obtain

$$x = \sqrt{\left(\frac{(ac+bd)\ (ad+bc)}{ab+cd}\right)}, y = \sqrt{\left(\frac{(ac+bd)\ (ab+cd)}{ad+bc}\right)}.$$

But according to the preceding Problem, the radius of the circle circumscribing the triangle ABC, whose sides are a, b, x, may be expressed by the formula $z = \frac{abx}{\sqrt{[4a^2b^2-(a^2+b^2-x^2)^2]}}$. In place of x, substitute the value just found for it, and decompose the result into factors; we shall have

$$z = \sqrt{\left[\frac{(ac+bd)\ (ad+bc)\ (ab+cd)}{(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)}\right]}.$$

This being proved, the area of the triangle $ABC = \frac{\frac{1}{4}abx}{z}$, that of the triangle $ADC = \frac{\frac{1}{4}cdx}{z}$; hence the area of the quadrilateral $ABCD =$

$\frac{1}{4}.\frac{(ab+cd)\ x}{z} =$

$\frac{1}{4}\sqrt{[(a+b+c-d)\ (a+b+d-c)\ (a+c+d-b)\ (b+c+d-a)]}$.

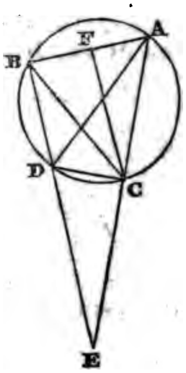
And if for the sake of brevity, we put $p = \frac{1}{2}\ (a+b+c+d)$, we shall have the area $ABCD = \sqrt{\ (p-a.p-b.p-c.p-d)}$. In fine, to obtain one of the angles, B for example, observe that the triangle ABC gives $\cos B = \frac{a^2+b^2-x^2}{2ab}$; substitute the value of x, and by reduction, you will have $\cos B = \frac{a^2+b^2-c^2-d^2}{2ab+2cd}$. Hence we obtain

$$\frac{1-\cos B}{1+\cos B} \text{ or } \text{tang.}^2 \tfrac{1}{2} B = \frac{(c+a)^2-(a-b)^2}{(a+b)\ -(c-d)^2} =$$

$$\frac{(a+c+d-b)\,(b+c+d-a)}{(a+b+c-d)\,(a+b+d-c)}. \quad \text{Hence tang. } \tfrac{1}{2}B = \sqrt{\left(\frac{p-a\,.\,p-b}{p-c\,.\,p-d}\right)}.$$

PROBLEM III.

In the quadrilateral ABCDE, *of which the opposite angles* B *and* C *are right, the two sides* AB, AC *with their contained angle* BAC *being given, to find the other two sides and the diagonal* AD.



Suppose $AC=b$, $AB=c$, and the angle $BAC=A$; if BD and AC are produced till they meet in E, the triangle BAE, right-angled at B, wherein are known the angle BAE and the side AB, will give $AE = \frac{c}{\cos A}$; hence $CE = \frac{c}{\cos A} - b$. Then the triangle DCE, right-angled at C, wherein are known the side CE and the angle $CDE = A$, will give $CD = CE \cot A = \frac{c - b\cos A}{\sin A}$. In like manner we shall have $BD = \frac{b - \cos A}{\sin A}$. These are the values of the two required sides of the quadrilateral.

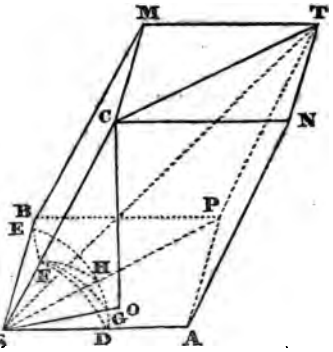
From this will be found $AD = \sqrt{(AC^2 + DC^2)} = \sqrt{\left(b^2 + \left(\frac{c - b\cos A}{\sin A}\right)^2\right)} = \frac{\sqrt{(b^2+c^2-2bc\cos A)}}{\sin A}$. But from the triangle BAC, we have $BC = \sqrt{(b^2+c^2-2bc\cos A.)}$ Hence the diagonal AD, which joins the two oblique angles, is to the diagonal BC, which joins the two right angles, as 1 is to sin A.

Scholium. The diagonal AD, is also the diameter of the circle in which the quadrilateral ABCD would be inscribed.

In this circle we should have the angle ABC = ADC; hence drawing CF perpendicular to AB, the triangles BFC, ADC are similar, and give AD : BC : : AC : FC : : 1 : sin A; which agrees with the result obtained above.

PROBLEM IV.

Given the three edges of a parallelepipedon, with the angles which they form together, to find the solidity of the parallelepipedon.



Suppose the edges SA $= f$, SB $= g$, SC $= h$; and the included angles ASB $= \gamma$, ASC $= \beta$, BSC $= \alpha$. If from the point C, a straight line CO be drawn at right angles to the plane ASB, the right-angled triangle CSO, will give CO $=$ CS sin CSO $= h$ sin CSO. Likewise the surface of the parallelogram ASBP $= fg \sin \gamma$. Hence, if we put S for the solidity of the parallelepipedon ST, we shall have S $= fgh \sin \alpha$ sin CSO. We have yet to find sin CSO.

To effect this, from the point S as a centre, with a radius $= 1$, describe a spherical surface meeting the straight lines SA, SB, SC, SO, in the points D, E, F, G; you will have a triangle DEF, in which the arc FG is perpendicular to ED, since the plane CSO is perpendicular to ASB. Now the triangle DEF, which has its three sides DE $= \gamma$, DF $= \beta$, EF $= \alpha$, gives cos DE $= \alpha$, DF $= \beta$, EF $= \gamma$, gives $\cos E = \dfrac{\cos \beta - \cos \alpha \cos \gamma}{\sin \alpha \sin \gamma}$ and

$$\sin E = \frac{\sqrt{(1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma)}}{\sin \alpha \sin \gamma}.$$

Then the right-angled triangle EFG gives sin GF, or sin CSO $=$ sin E sin EF $= \sin \gamma \sin E$. Hence S $= fgh \sin \alpha \sin \gamma \sin E$, or
S $= fgh \sqrt{(1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma)}$.

In this expression, the quantity under the radical is the product of the two factors, $\sin \alpha \sin \gamma + \cos \beta - \cos \alpha \cos \gamma$, and $\sin \alpha \sin \gamma - \cos \beta + \cos \alpha \cos \gamma$. The first $= \cos \beta - \cos (\alpha + \gamma) = 2 \sin \dfrac{\alpha + \beta + \gamma}{2} \sin \dfrac{\alpha + \gamma - \beta}{2}$, the second $= \cos(\alpha - \gamma) - \cos \beta = 2 \sin \dfrac{\alpha + \beta - \gamma}{2} \sin \dfrac{\beta + \gamma - \alpha}{2}$. Hence the required solidity

$$S = 2fgh \sqrt{\left[\sin \frac{\alpha + \beta + \gamma}{2} \sin \frac{\alpha + \beta - \gamma}{2} \sin \frac{\alpha + \gamma - \beta}{2} \sin \frac{\beta + \gamma - \alpha}{2}\right]}.$$

PROBLEM V.

The same things being given as in the last Problem, to find the expression of the diagonal which joins two opposite vertices.

Suppose the diagonal of the base SP $= z$, and the required diagonal ST $= u$: the triangle ASP, in which cos SAP $= -\cos \gamma$, will give

$z^2 = f^2 + g^2 + 2fg \cos\gamma$; likewise the triangle TSP, in which $\text{TPS} = -\cos \text{CSP}$, will give $u^2 = z^2 + h^2 + 2hz \cos \text{CSP}$. We now want only to find the cosine of the angle CSP, or of the arc FH: and here, in the spherical triangle EFH, we have $\cos \text{FH} = \cos \text{EF} \cos \text{EH} + \sin \text{EF} \sin \text{EH} \cos \text{E}$; substituting the values $\text{EF} = \alpha$ and $\cos \text{E} = \frac{\cos\beta - \cos\alpha \cos\gamma}{\sin\alpha \sin\gamma}$, we obtain $\cos \text{FH} = \cos\alpha \cos \text{EH} + \frac{\sin \text{EH}}{\sin\gamma}$ $(\cos\beta - \cos\alpha \cos\gamma) = \frac{\sin \text{EH} \cos\beta}{\sin\gamma} + \frac{\sin(\gamma - \text{EH}).\cos\alpha}{\sin\gamma} = \frac{\sin \text{EH} \cos\beta + \sin \text{DH} \cos\alpha}{\sin\gamma}$. Hence $2hz \cos \text{FH}$, or $2hz \cos \text{CSP} = 2h\cos\beta.\frac{z \sin \text{EH}}{\sin\gamma} + 2h\cos\alpha.\frac{z \sin \text{DH}}{\sin\gamma}$. But in the triangle BSP we have $\text{BP} = \frac{\text{SP} \sin \text{BSP}}{\sin \text{SBP}}$, and $\text{BS} = \frac{\text{SP} \sin \text{BPS}}{\sin \text{SBP}}$, which gives $\frac{z \sin \text{EH}}{\sin\gamma} = f$, and $\frac{z \sin \text{DH}}{\sin\gamma} = g$. Hence $2hz \cos \text{CSP} = 2fh \cos\beta + gh \cos\alpha$. Hence, finally, the square of the required diagonal $u^2 = f^2 + g^2 + h^2 + 2fg\cos\gamma + 2fh\cos\beta + 2gh\cos\alpha$.

Cor. The solid angle A, is formed by the edges f, g, h, making with each other, two by two, the angles $200° - \gamma$, $200° - \beta$, α; hence it is enough to change the signs of $\cos\gamma$ and $\cos\beta$ in the expression of SE^2, in order to have the expression of AM^2. By doing the same for the other two diagonals, we shall have the values of their squares as follows:

$$\text{ST}^2 = f^2 + g^2 + h^2 + 2fg\cos\gamma + 2fh\cos\beta + 2gh\cos\alpha$$
$$\text{AM}^2 = f^2 + g^2 + h^2 - 2fg\cos\gamma - 2fh\cos\beta + 2gh\cos\alpha$$
$$\text{BN}^2 = f^2 + g^2 + h^2 - 2fg\cos\gamma + 2fh\cos\beta - 2gh\cos\alpha$$
$$\text{CP}^2 = f^2 + g^2 + h^2 + 2fg\cos\gamma - 2fh\cos\beta - 2gh\cos\alpha$$

Hence we derive $\text{ST}^2 + \text{AM}^2 + \text{BN}^2 + \text{CP}^2 = 4f^2 + 4g^2 + h^2$

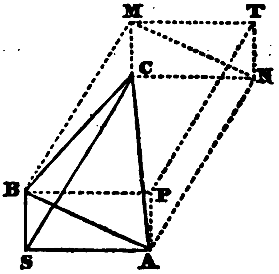
Hence, *in every parallelepipedon, the sum of the squares of the four diagonals, is equal to the sum of the squares of the twelve edges.* This remarkable theorem is analogous to that concerning the diagonals of a parallelogram (14. III), and might be derived immediately from it. For by means of the parallelograms SCTP, ABMN, we have

$$\text{ST}^2 + \text{CP}^2 = 2\,\text{SC}^2 + 2\,\text{SP}^2$$
$$\text{AM}^2 + \text{BN}^2 = 2\,\text{BM}^2 + 2\,\text{AB}^2.$$

Add these two equations together, observing that we have $\text{SC} = \text{BM}$, and $\text{SP}^2 + \text{AB}^2 = 2\,\text{SA}^2 + 2\,\text{SB}^2$; there will result $\text{ST}^2 + \text{AM}^2 + \text{BN}^2 + \text{CP}^2 = 4\,\text{SA}^2 + 4\,\text{SB}^2 + 4\,\text{SC}^2$.

PROBLEM VI.

Given the three edges ending at one vertex of a triangular pyramid, and the three angles which these edges form together, to find the solidity of the pyramid.



Let SABC be the proposed triangular pyramid, in which are given the edges $SA = f$, $SB = g$, $SC = h$, and the contained angles $ASB = \gamma$, $ASC = \beta$, $BSC = \alpha$. If on the edges SA, SB, SC, which are given in magnitude and position, the parallelepipedon ST be described, the pyramid, which is the third of the prism, will be the sixth of this parallelepipedon ST. Hence, calling P the solidity of the pyramid, by Prob. IV. we shall have

$$P = \tfrac{1}{6} fgh \sqrt{(1 - \cos^2\alpha - \cos^2\beta - {}^2\gamma + 2\cos\alpha\cos\beta\cos\gamma)} \text{ or}$$

$$P = \tfrac{1}{3} fgh \sqrt{\left[\sin\frac{\alpha+\beta+\gamma}{2} \sin\frac{\alpha+\beta-\gamma}{2} \sin\frac{\alpha+\gamma-\beta}{2} \sin\frac{\gamma+\beta-\alpha}{2}\right]}.$$

PROBLEM VII.

Given the six sides or edges of a triangular pyramid, to find its solidity.

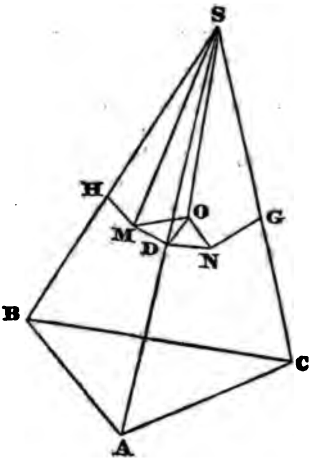
Retaining the same denominations as in the foregoing Problem, and, farther, putting $BC = f'$, $CA = g'$, $BA = h'$, we shall have $\cos\gamma = \frac{f^2 + g^2 - h'^2}{2fg}$, $\cos\beta = \frac{f^2 + h^2 - g'^2}{2fh}$ $\cos\alpha = \frac{g^2 + h^2 - f'^2}{2gh}$, substituting these values in the formula already found, and for the sake of abridgment, making $g^2 + h^2 - f'^2 = F$, $f^2 + h^2 - g'^2 = G$, $f^2 + g^2 - h'^2 = H$, we shall have the solidity $P = \frac{1}{12}\sqrt{(4f^2g^2h^2 - f^2F^2 - g^2G^2 - h^2H^2 + FGH)}$.

In the application of these formulas, it will be observed, that f', g', h', designate the sides of one face or base, while f, g, h, designate the other three edges which end at the vertex, their arrangement being such, that f is opposite to f', g to g', and h to h'.

Scholium. Let A be the sum of the four triangles composing the surface of the pyramid; let r be the radius of the inscribed sphere: it is easy to see that $P = A \times \frac{1}{3} r$; for the pyramid may be conceived as divided into four others, having for their common vertex the centre of the sphere, and for their bases the different faces of the pyramid. Hence we have the radius of the inscribed sphere $r = \frac{3P}{A}$.

PROBLEM VIII.

Given the same things as in Prob. 6., to find the radius of the sphere circumscribed about the pyramid.



Let M be the centre of the circle circumscribed about the triangle SAB; MO the perpendicular drawn through the point M to the plane SAB: in like manner, let N be the centre of the circle circumscribed about the triangle SAC; NO the perpendicular raised through the point N to the plane SAC. These two perpendiculars, lying in the same plane MDN perpendicular to SA, must intersect each other in a point O, which will be the centre of the circumscribed sphere; for the point O, as belonging to the perpendicular MO, is equally distant from the three points S, B, A; and this same point, as belonging to the perpendicular NO, is equally distant from the three points S, A, C: hence it is equally distant from the four points S, A, B, C.

The point M in the plane SAB may be conceived as determined by means of the quadrilateral SDMH, whose two angles D and H are right, and in which we have SD $= \frac{1}{2}f$, SH $= \frac{1}{2}g$, and ASB $= \gamma$. Hence (Prob. 3.) we shall have DM $= \frac{\frac{1}{2}g - \frac{1}{2}f\cos\gamma}{\sin\gamma}$; in like manner we shall have DN $= \frac{\frac{1}{2}h - \frac{1}{2}f\cos\beta}{\sin\beta}$.

Let us designate the angle MDN, which measures the inclination of the two planes SAB, SAC, by D; in the spherical triangle of which α, β, γ are the sides D will be the angle opposite the side α; and thus we shall have D $= \frac{\cos\alpha - \cos\gamma\cos\beta}{\sin\gamma\sin\beta}$; so that the angle D may be regarded as known.

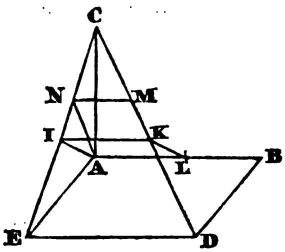
This being granted, in the quadrilateral OMDN, in which the two angles M and N are right, the two sides MD, DN, and the contained angle MDN = D being known, we shall have (Prob. 3.) the square of the diagonal $\text{OD}^2 = \frac{\text{DM}^2 + \text{DN}^2 - 2\,\text{DM} \times \text{DN}\cos\text{D}}{\sin^2\text{D}}$. Then in the triangle OSD right-angled at D, we shall have $\text{SO}^2 = \text{OD}^2 + \text{SD}^2$; which is the value of the square of the circumscribed sphere's radius.

If it is required to find the expression of the radius SO immediately by means of the data in Prob. 6., substitute the values of DM, DN, and next the values of *cos* D and of *sin* D; the result will be

$$SO=\tfrac{1}{2}\sqrt{\left\{\frac{f^2\sin^2\alpha+g^2\sin^2\beta+h^2\sin^2\gamma-2fg(\cos\gamma-\cos\beta\cos\alpha)-2fh(\cos\beta-\cos\alpha\cos\gamma)-2gh(\cos\alpha-\cos\gamma\cos\beta}{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma}\right\}}.$$

NOTE VI.

On the shortest distance between two straight lines not situated in the same plane.



Let AB, CD be the two straight lines, not in the same plane, whose distance it is required to find.

Along AB, extend two planes perpendicular to each other, and meeting CD, the one in C, the other in D; from the points C and D, draw CA and DB perpendicular to AB; in the plane ABD, draw DE parallel and AE perpendicular to BA, thus forming the rectangle ABDE; in the plane CAE, join CE, and draw AI perpendicular to CE; lastly, in the plane CDE, draw IK parallel to DE till it meet CD in K, make AL = IK, and join KL: then, *first*, the straight line KL will be at once perpendicular to the two given straight lines AB, CD; *secondly*, this straight line KL will be shorter than any other that can join two points in the lines AB, CD, the line KL, or its equal AI, thus being the shortest distance required.

First. The three straight lines AB, AC, AE being perpendicular to each other by construction, one of them AB is perpendicular to the plane of the other two; hence AB is perpendicular to AI: besides KI is parallel to DE, and DE to AB, hence KI is parallel to AB; and since we made AL = KI, the figure AIKL must be a rectangle. This being proved, the angle AIK is right as well as AIC; hence the straight line AI is perpendicular to the plane KIC or CDE; hence its parallel KL is perpendicular to the same plane CDE, and consequently perpendicular to CD. Hence, *first*, the straight line KL is at once perpendicular to the two straight lines AB, CD.

Secondly. Let M be any point in the straight line CD; if through this point, MN is drawn parallel to DE or AB, the distance of the point M from the straight line AB will be equal to AN, the angle BAN being right. Now we have AN $>$ AI; hence AI is the shortest distance of the given lines AB, CD.

Put the perpendiculars CA $= a$, and DB $=$ AE $= b$; we shall have CE $= \sqrt{(a^2+b^2)}$; and since the area of the triangle ACE is expressed equally by $\frac{1}{2}$ AC × AE and by $\frac{1}{2}$ CE × AI, we shall have $AI = \frac{AC \times EA}{CE} = \frac{ab}{\surd(a^2+b^2)}$. This latter is the expression for the shortest distance between the given lines.

If at the same time we put $AB = c$, and give the name A to the angle included between the two given lines, that is, to the angle CDE included between the line CD and DE a parallel to AB, the triangle CDE, right-angled at E, will give us $\cos CDE = \frac{DE}{CD}$, or $\cos A = \frac{c}{\surd(a^2+b^2+c^2)}$; because we have $CD^2 = CE^2 + ED^2 = a^2 + b^2 + c^2$. From this also we might infer that $\sin A = \frac{\surd(a^2+b^2)}{\surd(a^2+b^2+c^2)}$, and $\cot A = \frac{c}{\surd(a^2+b^2)}$.

NOTE VII.

On symmetrical Polyedrons.

It was for the sake of simplicity that, in Definition 16. VI., we supposed the plane to which the symmetrical polyedrons are referred to be the plane of a face: we might have supposed it to be any plane whatever; and the Definition in that case would have become more general without causing any change in the demonstration of Prop. 2., by which the mutual relations of the two polyedrons were established. A very correct idea of the nature of those two solids may also be obtained, by considering the one of them as the image of the other formed in a plane mirror, which mirror will occupy the place of the plane we have just been describing.

NOTE VIII.

On Proposition 25. *Book* VII.

This theorem, which Euler first proved, in the Memoirs of Petersburgh, *anno* 1758, presents several consequences worthy of being developed.

First. Let a be the number of triangles, b the number of quadrilaterals, c the number of pentagons, &c., composing the surface of a polyedron; the total number of faces will be $a+b+c+d+$ &c.; and the total number of their sides will be $3a+4b+5c+6d+$ &c. This latter number is twice that of the edges, since the same edge belongs at once to two faces; hence we shall have

$$H = a+b+c+d+ \text{\&c.}$$

$$2A = 3a+4b+5c+6d+ \text{\&c.}$$

And since, by the theorem in question, $S+H = A+2$, we obtain

$$2S = 4+a+2b+3c+4d+ \text{\&c.}$$

The first thing which strikes us in these values is, that the number of faces having an odd number of sides is always even.

For the sake of brevity, we may put $\omega = b+2c+3d+$ &c.; we shall then have

$$A = \tfrac{3}{2}H+\tfrac{1}{2}\omega,$$

$$S = 2+\tfrac{1}{2}H+\tfrac{1}{2}\omega.$$

Thus in every polyedron, we have always $A > \frac{3}{2}H$, and $S > 2+\frac{1}{2}H$, it being observed that the sign $>$ does not exclude equality if we should ever have $\omega = o$.

The number of all the plane angles in the polyedron is $2A$, that of the solid angles is S; so that the mean number of plane angles which go to the formation of a solid angle is $\dfrac{2A}{S}$.

This number cannot be less than 3, because at least three plane angles are required to form a solid angle; hence we must have $2A > 3S$, the sign $>$ not excluding equality. If in place of A and S, we substitute their values in terms of H and ω, we shall have $3H+\omega > 6+\frac{3}{2}H+\frac{3}{2}\omega$, or $3H > 12+\omega$. Bring back the values of H and ω into terms of a, b, c, &c., we shall have

$$3a+2b+c > 12+e+2f+3g+ \text{\&c.};$$

from which it appears that a, b, c cannot all be nothing at once, and that consequently there exists no polyedron all whose faces have each more than five sides.

Since we have $H > 4+\frac{1}{3}\omega$, by substituting the values of S and A, we get $S > 4+\frac{2}{3}\omega$, and $A > 6+\omega$. But at the same time, we have $\omega < 3H-12$; and from it there result $S < 2H-4$, and $A < 3H-6$, where it must be recollected that the signs $>$ and $<$ do not exclude equality. These limits are observed in all polyedrons generally.

Second. Suppose $2A > 4S$, which is true of a multitude of polyedrons, and particularly of those which have all their solid angles formed by four planes or more; we shall in this case have $H > 8+\omega$, or by substituting,

$$a > 8+c+2d+3e+ \text{\&c.}$$

Hence the solid must at least have eight triangular faces; the limit $H > 8+\omega$ gives $S > 6+\omega$, and $A > 12+2\omega$. But we have, at the same time, $\omega > H-8$; and from this there result $S > H-2$, and $A > 2H-4$.

Third. Suppose $2A > 5S$, which, among other polyedrons, includes all such as have each of their solid angles formed by five planes at least; there will result from it $H \not< 20+3\,\omega$, or

$$a \not< 20+2\,b+5\,c+8\,d+ \&c.$$

We shall at the same time have $S \not< 12+2\,\omega$, and $A \not< 30+5\,\omega$; and, lastly, from $\omega \not< \frac{1}{3}(H-20)$, we shall deduce the limits $S \not< \frac{2}{3}(H-2)$, and $A \not< \frac{5}{3}(H-2.)$

We cannot suppose $2\,A = 6\,S$: because we have $2\,A+2\,\omega+12=6\,S$ generally; hence there is no polyedron which has all its solid angles formed of six planes or more; and accordingly, the least value, which each plane angle, one with another, could have, would be the angle of an equilateral triangle, and six of such angles make four right angles, which is too much for a solid angle.

Fourth. Let us examine a polyedron whose faces are all triangular; we shall have $\omega = o$, which will give $A = \frac{3}{2}H$, and $S = 2+\frac{1}{2}H$. Suppose farther, that the solid angles of the polyedron are in part quintuple, in part sextuple; let p be the number of quintuple solid angles, q of sextuple; we shall have $S = p+q$ and $2\,A = 5\,p+6\,q$, which give $6\,S-2\,A = p$: but we have besides $A = \frac{3}{2}H$, and $S = 2+\frac{1}{2}H$; hence $p = 6\,S-2\,A = 12$. Hence *if a polyedron has all its faces triangular, and if its solid angles are in part quintuple, in part sextuple, the number of quintuple solid angles will always amount to* 12. The sextuple may be in any number whatever: thus, leaving q undetermined, we shall have all those solid angles $S = 12+q$, $H = 20+2\,q$, $A = 30+3\,q$.

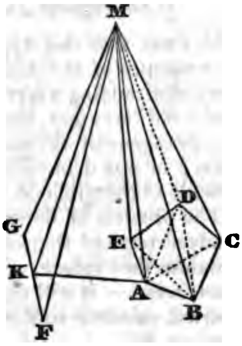
We shall finish these applications by investigating the number of conditions or data necessary for determining any polyedron; an interesting question, which does not yet seem to have been resolved.

Suppose, first, that the polyedron is *of a determinate kind,* in other words, that we know the number of its faces, the number of their sides individually, and their arrangement with regard to one another. We therefore know the numbers, H, S, A, and likewise a, b, c, d, &c.; we only want farther to discover the actual number of given quantities, lines or angles, by means of which the polyedron may be constructed and determined.

Let us examine one of the polyedron's faces, which we shall regard as its base. Suppose n to be the number of its sides; there will be $2\,n-3$ data required to determine this base. The solid angles out of this base amount in number to $S-n$: the vertex of each solid angle requires three data for determining it; hence the position of $S-n$ vertices will require $3\,S-3\,n$; to which adding the $2\,n-3$ data of the base, we shall have in all $3\,S-n-3$. But this number in general is too great; it must be diminished by the number of conditions necessary for making the vertices which correspond to the same face lie all in one plane. We have called the number of sides in the base n; let us in like manner call the number of sides in the other faces n', n'', &c. Three points determine a plane; hence whatever more than 3 are found in each of the numbers n', n'', &c., will give just so many conditions for making the different vertices lie in the planes of the faces to which they belong; and the total number of conditions will be equal to the sum $(n'-3)+(n''-3)+(n'''-3)+$&c.

But the number of terms in this series is $H-1$; and, moreover, $n+n'+n''+$ &c. $=2A$: hence the sum of the series will be $2A-n-3(H-1)$. From this sum, take away $3S-n-3$; there will remain $3S-2A+3H-6$, a quantity, which by reason of $S+H=A+2$, may be reduced to A. Hence *the number of data necessary for determining a polyedron, among all those of the same species, is equal to the number of its edges.*

Observe, however, that the data here spoken of must not be taken at random among the lines and angles which constitute the elements of the polyedron; for although there were as many equations as unknown quantities, it might happen that certain relations between the known quantities might render the problem indeterminate. Thus from the theorem just discovered, it might seem that a knowledge of the edges alone would be enough for determining the polyedron; yet there are cases in which this knowledge of itself is not sufficient. If, for example, any prism not triangular were given, an infinite number of other prisms might be formed having edges equal and placed in the same manner. For, whenever the base has more than three sides, the angles may be changed though the same sides are retained, and thus the base may have an infinite number of different forms; also the position of the prism's longitudinal edge with regard to the plane of the base may be changed; finally, these two changes may be combined with each other; and from every new arrangement, a new prism will result still having its edges or sides unchanged. From all which, it is clear, that in this case the edges alone are not enough for determining the solid.



The data which it is proper to select for determining a solid, are those which leave no indeterminateness, and give absolutely only one solution. And first, the base ABCDE will be determined by this among other modes; by knowing the side AB with the adjacent angles BAC, ABC for the point C; the angles BAD, ABD for the point D; and so for all the rest. Next, let M be a point without the base whose position it is required to determine: this point will be determined, if, imagining the pyramid MABC or simply the plane MAB, we know the angles MAB, ABM, and the inclination of the plane MAB to the base ABC. If by means of three analogous data, the position of each vertex lying without the base of the polyedron is determined, the polyedron, it is evident, will be absolutely determined, and so that two polyedrons constructed with the same data must of necessity be equal; or symmetrically equal, if constructed on different sides of the plane of the base.

It is not always required to have three data for determining each vertex of a polyedron; for if the point M must be found in a plane already determined, whose intersection with the base is FG, it will be sufficient, after having assumed FG at will, if we know the angles MGF, MFG; and thus one datum less will be enough. If the point M must be found in two planes already determined, or in their common intersection MK, which meets ABC in K, we shall in this case already know the side AK, the angle AKM, and the inclination of the plane AKM to the base; hence it will be enough to have for a new datum the angle MAK. By such means, the number of data necessary for determining a polyedron absolutely and without any ambiguity will always be reduced to A, the number of its edges.

The side AB and a number $A - 1$ of given angles determine a polyedron; another side assumed at pleasure and the same angles determined a similar polyedron. Hence it follows that *the number of conditions necessary for determining the similarity of two polyedrons belonging to the same species is equal to the number of edges* minus *one.*

The question we have just resolved would be much simpler, if, instead of knowing the species of the polyedron, we knew only S the number of its solid angles. In that case, determine three vertices at pleasure, by means of a triangle in which are three data; this triangle will be regarded as the base of the solid; then the number of vertices out of this base will be $S - 3$; and since the determination of each of them requires three data, the total number of data necessary for determining the polyedron will evidently be $3 + 3\,(S - 3)$, or $3\,S - 6$.

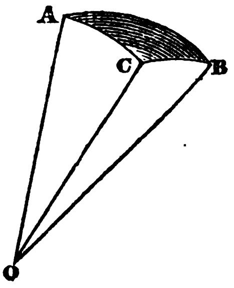
Hence $3\,S - 7$ conditions will be necessary for determining the similarity of two polygons having the same number S of solid angles.

NOTE IX.

On the regular Polyedrons. (*See Appendix to* Book VII.)

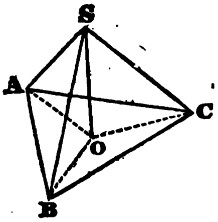
In Prop. 2 of the Appendix referred to, we set about demonstrating the existence of the five regular polyedrons, that is to say, the possibility of arranging a certain number of equal planes, so that a solid, uniform in all its extent, shall result. It appeared to us that, in other works, this arrangement was assumed as existing, without any good reason being given; or was demonstrated, as in Euclid, by figures too complex and difficult to comprehend.

The problem of determining the inclination of two adjacent faces in a polyedron, and that of determining the radii of the inscribed and the circumscribed sphere, are in Problems 3. and 4. reduced to very simple constructions: it will not, however, be without use to apply the trigonometrical calculus to these same Problems, and so to obtain some new propositions.

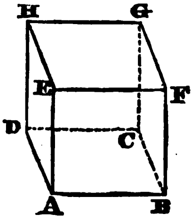


Let a, b, c be the three plane angles which form the solid angle O; and let it be proposed to find the inclination of the planes in which the angles a and b are situated. From the centre O, describe the spherical triangle ABC; its three sides $BC = a$, $AC = b$, $AB = c$ will be known: we are required to find the angle C included between the sides a and b. Now, by admitted formulas, we have $\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$.

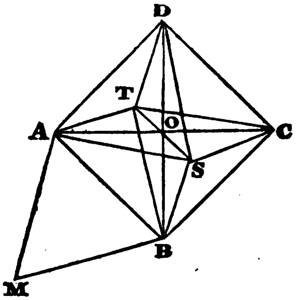
This formula, when applied to the five regular polyedrons, will shew us the inclination of two adjacent faces in each of these solids.



In the tetraedron, the three plane angles forming the solid angle S are angles of equilateral triangles: therefore, put the semicircumference or the arc of 200° $= \pi$; we shall have $a = b = c = \frac{1}{3}\pi$; hence $\cos C = \dfrac{\cos a - \cos^2 a}{\sin^2 a} = \dfrac{\cos a\,(1 - \cos a)}{1 - \cos^2 a} = \dfrac{\cos a}{1 + \cos a}$: but we already know that $\cos \frac{1}{3}\pi = \frac{1}{2}$; hence $\cos C = \frac{1}{3}$.



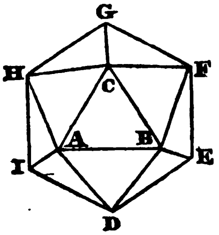
In the hexaedron or cube, the three plane angles forming the solid angle A are right angles: hence we have $a = b = c = \frac{1}{2}\pi$, and $\cos a = 0$; hence $\cos C = 0$. Hence the angle of two adjacent faces is right.

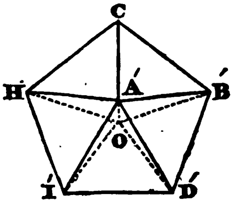


In the octaedron, if we put $a =$ DAS $= \frac{1}{3}\pi$, $b =$ DAT $= \frac{1}{3}\pi$, $c =$ TAS $= \frac{1}{2}\pi$, we shall have $\cos C = \dfrac{\cos \frac{1}{2}\pi - \cos^2 \frac{1}{3}\pi}{\sin^2 \frac{1}{3}\pi}$. But $\cos \frac{1}{2}\pi = 0$, $\cos \frac{1}{3}\pi = \frac{1}{2}$, $\sin \frac{1}{3}\pi = \frac{1}{2}\sqrt{3}$; hence $\cos C = -\frac{1}{3}$. From which it appears, that the inclination of two faces in the octaedron, and the inclination of two faces in the tetraedron, are two angles which are supplements of each other.

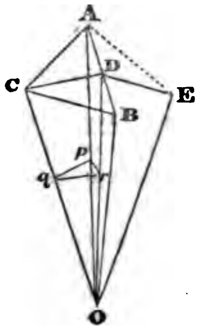
In the dodecaedron, a solid angle is formed of three plane angles, each equal to the angle of a regular pentagon; hence, making $a = b = c = \frac{3}{5}\pi$ we shall have $\cos C = \dfrac{\cos a}{1 + \cos a}$; but $\cos \frac{3}{5}\pi = -\sin \frac{1}{10}\pi = \dfrac{1 - \sqrt{5}}{4}$; hence $\cos C = \dfrac{1 - \sqrt{5}}{5 - \sqrt{5}}, = -\dfrac{1}{\sqrt{5}}$, $\sin C = \dfrac{2}{\sqrt{5}}$, and tang. $C = -2$.

In the icosaedron, we must make $c =$ C′B′D′ $= \frac{3}{5}\pi$, $a = b =$ C′ B′ A′ $= \frac{1}{3}\pi$; and we shall have $\cos C = \dfrac{\cos \frac{3}{5}\pi - \cos^2 \frac{1}{3}\pi}{\sin^2 \frac{1}{3}\pi} = \dfrac{\frac{1}{4}(1 - \sqrt{5}) - \frac{1}{4}}{\frac{3}{4}} = \dfrac{-\sqrt{5}}{3}$; hence $\sin C = \frac{2}{3}$.





Such are the very simple expressions, by means of which the inclination of two faces in all the five regular polyedrons is determined. We shall farther observe, that they might all have been included under one single formula.



Thus, let n be the number of sides in each face, m the number of plane angles combined in each solid angle; if from the point O with a radius $=1$, a spherical surface be described to meet the line OA, OC, OD in p, q, r, we shall have a spherical triangle pqr, in which are known the angle r, the angle $p = \frac{\pi}{m}$, and the angle $q = \frac{\pi}{n}$; hence by admitted formulas, we shall have $\cos qr = \frac{\cos p}{\sin q}$. But $\cos qr = \cos COD = \sin CDO$ $\sin \frac{1}{2} C$, C designating the angle CDE; hence $\sin \frac{1}{2} C = \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}}$; a general formula which, if applied successively to the five polyedrons, will give the same values of $\cos C$, or of $1 - \sin^2 \frac{1}{2} C$, as were found above by another method. For this purpose, we must substitute the values of m and n in each particular case, namely,

	Tetraedron.	Hexaedron.	Octaedron.	Dodecaedron.	Icosaedron.
$m =$	3	3	4	3	5
$n =$	3	4	3	5	3

The same spherical triangle pqr, from which the inclination of two adjacent faces has now been deduced, gives also $\cos pq = \cot p \cot q$, or $\frac{CO}{OA} = \cot \frac{\pi}{m} \cot \frac{\pi}{n}$. Hence, if the radius of the sphere circumscribed about the polyedron is called R, the radius of the sphere inscribed in the same polyedron being called r, we shall have $\frac{R}{r} = \tang \frac{\pi}{m} \tang \frac{\pi}{n}$; also, making $AB = a$, we have $CA = \frac{\frac{1}{2}a}{\sin \frac{\pi}{n}}$, and consequently $R^2 = r^2 + \frac{\frac{1}{4}a^2}{\sin^2 \frac{\pi}{n}}$. These two equations will give, for each polyedron, the values of R and r, the radii of the circumscribed and of the inscribed sphere. Likewise, supposing C known, we have $r = \frac{1}{2}a \cot \frac{\pi}{n} \tang \frac{1}{2}C$, and $R = \frac{1}{2}a \tang \frac{\pi}{m} \tang \frac{1}{2}C$.

In the dodecaedron and the icosaedron, the ratio $\frac{R}{r}$ has evidently the same value, tang $\frac{\pi}{3}$ tang $\frac{\pi}{5}$. Hence if R is the same for both, r also will be the same; in other words, if these two solids are inscribed in the same sphere, they will also be circumscribed about the same sphere, and *vice versa*. The same property subsists between the hexaedron and the octaedron, the value of $\frac{R}{r}$ being tang $\frac{\pi}{3}$ tang $\frac{\pi}{4}$, for both.

We may observe, that the regular polyedrons are not the only solids contained by equal and regular polygons. If two regular tetraedrons are adjusted to each other by a common face, a solid bounded by six equal and equilateral triangles will result. Another solid might be formed with ten equal and equilateral triangles; but the regular polyedrons are the only ones which at the same time have their solid angles equal.

NOTE X.

On the Area of the Spherical Triangle.

LET 1 be the radius of the sphere, π the semicircumference of a great circle; let a, b, c be the three sides of a spherical triangle; A, B, C the arcs of great circles which measure the opposite angles. Put $A+B+C-\pi=S$. According to what was proved in the text (Prop. 23. VII.), the area of the spherical triangle is equal to the arc S multiplied by the radius, and consequently it is represented by S. Now, by *Napier's* analogies, we have,

$$\text{tang}\,\frac{A+B}{2} : \cot\frac{C}{2} :: \cos\frac{a-b}{2} : \cos\frac{a+b}{2};$$

from which having found the value of tang $\frac{1}{2}(A+B)$, we shall easily deduce that of tang $(\frac{1}{2}A+\frac{1}{2}B+\frac{1}{2}C)=-\cot\frac{1}{2}S$: we shall thus have

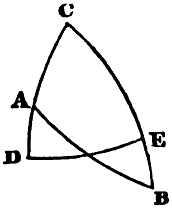
$$\cot\tfrac{1}{2}S=\frac{\cot\frac{1}{2}a\cot\frac{1}{2}b+\cos C}{\sin C},$$

a very simple formula, which will serve for calculating the area of a spherical triangle, when two sides a, b, and the included angle C, are known. Some remarkable consequences may also be deduced from it.

First. If the angle C is constant, and likewise the product $\cot\frac{a}{2}\cot\frac{b}{2}$, the area of the spherical triangle represented by S will also be constant. Hence two triangles CAB, CDE, which have an equal angle C, will be equivalent, if they have

$$\text{tang } \tfrac{1}{2}\,\text{CA} : \text{tang } \tfrac{1}{2}\,\text{CD} :: \text{tang } \tfrac{1}{2}\,\text{CE} : \text{tang } \tfrac{1}{2}\,\text{CB},$$

in other words, if the tangents of the half-sides containing the equal angles are reciprocally proportional.



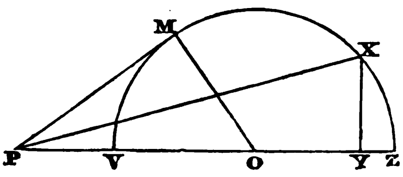
Second. On the given side CD, and with the same angle C, to make a triangle CDE equivalent to the given triangle CAB, we must determine CE by the proportion

$$\text{tang } \tfrac{1}{2}\text{CE} : \text{tang } \tfrac{1}{2}\text{CA} :: \text{tang } \tfrac{1}{2}\text{CB} : \text{tang } \tfrac{1}{2}\text{CE}.$$

Third. With the angle C at the vertex, to make an isosceles triangle DCE equivalent to the given triangle CAB, we must take tang $\tfrac{1}{2}$CD, or tang $\tfrac{1}{2}$CE, a mean proportional between tang $\tfrac{1}{2}$CA and tang $\tfrac{1}{2}$CB.

Fourth. The same formula $\cot \tfrac{1}{2}S = \dfrac{\cot \frac{1}{2} a \cot \frac{1}{2} b + \cos C}{\sin C}$ will afford a very simple demonstration of Prop. 20. Book VII., where it is proved that of all the spherical triangles formed with two given sides *a* and *b*, the greatest is the one in which the angle C, included between the given sides, is equal to the sum of the other two angles A and B.

With the radius $OZ = 1$, describe the semicircumference VMZ; make the arc $ZX = C$; and on the other side of the centre, take $OP = \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b$; lastly join PX, and draw XY perpendicular to PZ.



In the right-angled triangle PXY, we have $\cot P = \dfrac{PY}{XY} = \dfrac{\cot \frac{1}{2}a \cot \frac{1}{2}b + \cos C}{\sin C}$; hence $P = \tfrac{1}{2}S$; hence the surface S will be a *maximum*, when the angle P is one. Now it is evident that if PM is drawn to touch the circumference, the angle MPO will be the *maximum* of the angles P; in which case, we shall have $MPO = MOZ - \tfrac{1}{2}\pi$. Hence the spherical triangle with two given sides, will be a *maximum*, if we have $\tfrac{1}{2}S = C - \tfrac{1}{2}\pi$, or $C = A + B$; which agrees with the Proposition referred to.

From this construction, it is also plain that no *maximum* could exist, if the point P were within the circle, that is, if we had $\cot \tfrac{1}{2}a \cot \tfrac{1}{2}b \angle 1$; a condition from which we successively deduce $\cot \tfrac{1}{2} a \angle \text{tang } \tfrac{1}{2} b$, $\text{tang } (\tfrac{1}{2}\pi - \tfrac{1}{2}a) \angle \text{tang } \tfrac{1}{2}b$, $\tfrac{1}{2}\pi - \tfrac{1}{2}a \angle \tfrac{1}{2}b$, and finally $\pi \angle a + b$; which also agrees with the Scholium of the same Proposition.

Problem I. *To find the surface of a spherical triangle by means of its three sides.*

To effect this, in the formula

$$\cot \tfrac{1}{2}S = \frac{\cot \tfrac{1}{2}a \cot \tfrac{1}{2}b + \cos C}{\sin C},$$

we must substitute the values of sin C and cos C expressed in terms of a, b, c. Now we have $\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$ and $\cot \tfrac{1}{2}a \cot \tfrac{1}{2}b = \frac{1 + \cos a}{\sin a} \cdot \frac{1 + \cos b}{\sin b}$; hence results

$$\cos C + \cot \tfrac{1}{2}a \cot \tfrac{1}{2}b = \frac{1 + \cos a + \cos b + \cos c}{\sin a \sin b}.$$

Next, the value of cos C gives

$$1 + \cos C = \frac{\cos c - \cos (a+b)}{\sin a \sin b} = \frac{2 \sin \frac{a+b+c}{2} \cdot \sin \frac{a+b-c}{2}}{\sin a \sin b};$$

$$1 - \cos C = \frac{\cos (a-b) - \cos c}{\sin a \sin b} = \frac{2 \sin \frac{a+c-b}{2} \cdot \sin \frac{b+c-a}{2}}{\sin a \sin b}.$$

Multiply these two quantities together, and extract the root of the product; we shall have

$$\sin C = \frac{2\sqrt{\left(\sin \frac{a+b+c}{2} \cdot \sin \frac{a+b-c}{2} \cdot \sin \frac{a+c-b}{2} \cdot \sin \frac{b+c-a}{2}\right)}}{\sin a \sin b}.$$

Hence, finally,

$$\cot \tfrac{1}{2} S = \frac{1 + \cos a + \cos b + \cos c}{2\sqrt{\left(\sin \frac{a+b+c}{2} \cdot \sin \frac{a+b-c}{2} \cdot \sin \frac{a+c-b}{2} \cdot \sin \frac{b+a-c}{2}\right)}}$$

This formula resolves the proposed Problem; but a more simple result may be obtained.

For this purpose, let us resume the formula,

$$\cot \tfrac{1}{2} S = \frac{\cot \tfrac{1}{2} a \cot \tfrac{1}{2} b + \cos C}{\sin C},$$

we first deduce from it $1 + \cot^2 \tfrac{1}{2} S$ or $\frac{1}{\sin^2 \tfrac{1}{2} S} = \frac{\cot^2 \tfrac{1}{2} a \cot^2 \tfrac{1}{2} b + 2 \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cos C + 1}{\sin^2 C}$. Now, the value of cos C gives $2 \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cos C = \frac{\cos c - \cos a \cos b}{2 \sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b}$. In the numerator, instead of cos c, cos a, cos b, insert their values, 1 —

$2\sin^2\frac{1}{2}c$, $1-2\sin^2\frac{1}{2}a$, $1-2\sin^2\frac{1}{2}b$; by reduction, we shall have

$$2\cot\tfrac{1}{2}a\cot\tfrac{1}{2}b\cos C=\frac{\sin^2\frac{1}{2}a+\sin^2\frac{1}{2}b-\sin^2\frac{1}{2}c}{\sin^2\frac{1}{2}a\sin^2\frac{1}{2}b}-2.$$

Besides, we have $\cot^2\frac{1}{2}a.\cot^2\frac{1}{2}b=\frac{1-\sin^2\frac{1}{2}a}{\sin^2\frac{1}{2}a}.\frac{1-\sin^2\frac{1}{2}b}{\sin^2\frac{1}{2}b}=$ $\frac{1-\sin^2\frac{1}{2}a-\sin^2\frac{1}{2}b}{\sin^2\frac{1}{2}a\sin^2\frac{1}{2}b}+1$. Hence, substituting these values, we shall have $\frac{1}{\sin^2\frac{1}{2}S}=\frac{1-\sin^2\frac{1}{2}c}{\sin^2\frac{1}{2}a\sin^2\frac{1}{2}b\sin^2 C}$, which gives $\sin\frac{1}{2}S=$ $\frac{\sin\frac{1}{2}a\sin\frac{1}{2}b\sin C}{\cos\frac{1}{2}c}$; and resuming the value of sin C, we have

$$\sin\tfrac{1}{2}S=\frac{\sqrt{\left(\sin\frac{a+b+c}{2}.\sin\frac{a+b-c}{2}.\sin\frac{a+c-b}{2}.\sin\frac{b+c-a}{2}\right)}}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}.$$

A formula convenient for logarithmic calculations.

Multiply it by the value of $\cot\frac{1}{2}S$; there will result

$$\cos\tfrac{1}{2}S=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}=\frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}.$$

A new formula, which has the advantage of being composed of rational terms.

From this we deduce farther $\frac{1-\cos\frac{1}{2}S}{\sin\frac{1}{2}S}$ or

$$\text{tang}\,\tfrac{1}{4}S=\frac{1-\cos^2\frac{1}{2}a-\cos^2\frac{1}{2}b-\cos^2\frac{1}{2}c+2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}{\sqrt{\left(\sin\frac{a+b+c}{2}.\sin\frac{a+b-c}{2}.\sin\frac{a+c-b}{2}.\sin\frac{b+c-a}{2}\right)}}.$$

But the numerator of this expression may be put under the form $(1-\cos^2\frac{1}{2}a)(1-\cos^2\frac{1}{2}b)-(\cos\frac{1}{2}a\cos\frac{1}{2}b-\cos\frac{1}{2}c)^2$, which may be decomposed (as in Prob. 4. Note V.) into two factors, namely,

$\sin\frac{1}{2}a\sin\frac{1}{2}b+\cos\frac{1}{2}a\cos\frac{1}{2}b-\cos\frac{1}{2}c$ and $\sin\frac{1}{2}a\sin\frac{1}{2}b-\cos\frac{1}{2}a\cos\frac{1}{2}b+\cos\frac{1}{2}c$; these are ultimately reducible, the first to $\cos(\frac{1}{2}a-\frac{1}{2}b)-\cos\frac{1}{2}c=2\sin\frac{a+c-b}{4}.\sin\frac{b+c-a}{4}$, the second to $\cos\frac{1}{2}c-\cos(\frac{1}{2}a+\frac{1}{2}b)=2\sin\frac{a+b+c}{4}\sin\frac{a+b-c}{4}$. Hence

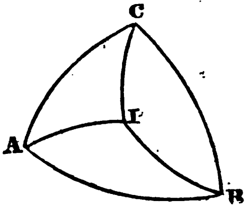
$$\text{tang}\,\tfrac{1}{4}S=\frac{4\sin\frac{a+b+c}{4}.\sin\frac{a+b-c}{4}.\sin\frac{a+c-b}{4}.\sin\frac{b+c-a}{4}}{\sqrt{\left(\sin\frac{a+b+c}{2}\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}\sin\frac{b+c-a}{2}\right)}}.$$

But we have $\frac{\sin\frac{1}{2}p}{\sqrt{\sin p}}=\sqrt{\left(\frac{\sin^2\frac{1}{2}p}{2\sin\frac{1}{2}p\cos\frac{1}{2}p}\right)}=\sqrt{(\frac{1}{2}\,\text{tang}\,\frac{1}{2}p)}$; hence finally

$$\operatorname{tang}\tfrac{1}{4}S=\surd\left(\operatorname{tang}\frac{a+b+c}{4}\operatorname{tang}\frac{a+b-c}{4}\operatorname{tang}\frac{a+c-b}{4}\operatorname{tang}\frac{b+a-c}{4}\right).$$

For this very elegant formula, we are indebted to Simon Lhuillier.

Problem ii. *Given the three sides* BC $=$ a, AC $=$ b, AB $=$ c, *to determine the position of the point* I, *the pole of the circle described about the triangle* ABC.



Put the angle ACI $= x$; the arc AI $=$ CI $=$ BI $= \varphi$; in the triangles CAI, CBI, by admitted formulas, we shall have

$$\cos x=\frac{\cos\varphi-\cos b\cos\varphi}{\sin b\sin\varphi}=\frac{1-\cos b}{\sin b}\cot\varphi$$

$$=\frac{\sin b}{1+\cos b}\cot\varphi,\ \cos(C-x)=\frac{1-\cos a}{\sin a}\cot\varphi.$$

Hence $\dfrac{\cos(C-x)}{\cos x}$, or $\cos C+\sin C\operatorname{tang}x=\dfrac{(1+\cos b)(1-\cos a)}{\sin a\sin b}$; substituting in this equation the values of cos C and sin C expressed in terms of a, b, c, and making, for the sake of abridgment,

$M=\surd\,(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)$; we shall deduce from it $\operatorname{tang}x=\dfrac{1+\cos b-\cos c-\cos a}{M}$; a formula which determines the angle ACI. It may be observed, that by reason of the isosceles triangles ACI, ABI, BCI, we have ACI $=\frac{1}{2}(C+A-B)$; we might also have ACI $=\frac{1}{2}(B+C-A)$, and BAI $=\frac{1}{2}(A+B-C)$. Hence result these remarkable formulas:

$$\operatorname{tang}\tfrac{1}{2}(A+C-B)=\frac{1+\cos b-\cos a-\cos c}{M}$$

$$\operatorname{tang}\tfrac{1}{2}(B+C-A)=\frac{1+\cos a-\cos b-\cos c}{M}$$

$$\operatorname{tang}\tfrac{1}{2}(A+B-C)=\frac{1+\cos c-\cos a-\cos b}{M}.$$

To these may be added the formula which gives cot $\frac{1}{2}$ S, and admits of being put under the form,

$$\operatorname{tang}\tfrac{1}{2}(A+B+C)=\frac{-\cos a-\cos b-\cos c}{M}.$$

The value of tang x just found, gives

$$1+\operatorname{tang}^2 x\text{ or }\frac{1}{\cos^2 x}=\frac{2(1+\cos b)(1-\cos c)(1-\cos a)}{M^2}$$

$$=\frac{16\cos^2\frac{1}{2}b\sin^2\frac{1}{2}c\sin^2\frac{1}{2}a}{M^2};\ \text{hence}$$

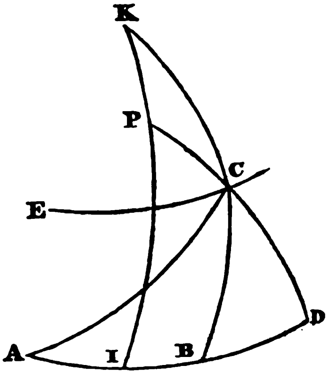
$\dfrac{1}{\cos x}=\dfrac{4\cos\frac{1}{2}b\sin\frac{1}{2}c\sin\frac{1}{2}a}{M}$. But from the equation

$\cos x = \frac{1 - \cos b}{\sin b} \cot \varphi = \text{tang} \tfrac{1}{2} b \cot \varphi$, we get $\text{tang}\, \varphi = \frac{\text{tang} \frac{1}{2} b}{\cos x}$;

hence $\text{tang}\, \varphi = \frac{4 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\text{M}}$

$$= \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sqrt{\left(\sin \frac{a+b+c}{2} \cdot \sin \frac{a+b-c}{2} \cdot \sin \frac{a+c-b}{2} \cdot \sin \frac{b+c-a}{2}\right)}}.$$

Problem iii. *On the surface of the sphere, to determine the line, in which are situated the vertices of all the triangles, having the same base, and the same surface.*



Let ABC be one of the spherical triangles, whose common base is AB $= \text{C}$, and given surface $\text{A} + \text{B} + \text{C} - \pi = \text{S}$. Let IPK be an indefinite perpendicular drawn from the middle of AB, then IP being taken equal to a quadrant, P will be the pole of the arc AB, and the arc PCD drawn through the points P, C will be perpendicular to AB. Put $\text{ID} = p$, $\text{CD} = q$; the right-angled triangles ACD, BCD, in which we have $\text{AC} = b$, $\text{BC} = a$, $\text{AD} = p + \frac{1}{2} c$, $\text{BD} = p - \frac{1}{2} e$, will give $\cos a = \cos q \cos (p - \frac{1}{2} c)$, $\cos b = \cos q \cos (p + \frac{1}{2} c)$. But we found above

$\cot \frac{1}{2} \text{S} = \frac{1 + \cos a + \cos b + \cos c}{\sin a \sin b \sin \text{C}}$:

substitute in this formula the values $\cos a + \cos b = 2 \cos q \cos p \cos \frac{1}{2} c$, $1 - \cos c = 2 \cos^2 \frac{1}{2} c$, $\sin b \sin \text{C} = \sin c \sin \text{B} = 2 \sin \frac{1}{2} c \cos c \sin \text{B}$; we shall have

$$\cot \tfrac{1}{2} \text{S} = \frac{\cos \frac{1}{2} c + \cos p \cos q}{\sin a \sin \frac{1}{2} c \sin \text{B}}$$

Moreover, in the right-angled triangle BCD, we have $\sin a \sin \text{B} = \sin q$; hence $\cot \frac{1}{2} \text{S} = \frac{\cos \frac{1}{2} c + \cos p \cos q}{\sin \frac{1}{2} c \sin q}$, or $\cos p \cos q = \cot \frac{1}{2} \text{S} \sin \frac{1}{2} c \sin q - \cos \frac{1}{2} c$.

This is the relation between p and q, which must determine the line whereon are situated, all the points C.

Having produced IP by a quantity $\text{PK} = x$, join KC, and put $\text{KC} = y$; in the triangle PKC, which has $\text{PC} = \frac{1}{2} \pi - q$, and the angle $\text{KPC} = \pi - p$, the side KC will be found by the formula $\cos \text{KC} = \cos \text{KPC} \sin \text{PK} \sin \text{PC} + \cos \text{PK} \cos \text{PC}$, or

$$\cos y = \sin q \cos x - \sin x \cos q \cos p;$$

in which, by substituting, in place of $\cos q \cos p$, its value $\cot \frac{1}{2} S \sin \frac{1}{2} c \sin q - \cos \frac{1}{2} c$, we shall have $\cos y = \sin x \cos \frac{1}{2} o + \sin q (\cos x - \sin x \cot \frac{1}{2} S \sin \frac{1}{2} c)$. Whence it appears that taking $\cos x - \sin x \cot \frac{1}{2} S \sin \frac{1}{2} c = 0$, or $\cot x = \cot \frac{1}{2} S \sin \frac{1}{2} c$, we shall have $\cos y = \sin x \cos \frac{1}{2} c$, and thus the value of y will become constant.

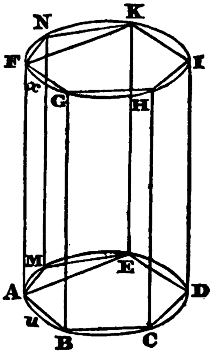
Hence, if after drawing the arc IP at right angles to the middle of the base AB, the part PK beyond the pole is taken, so that $\cot PK = \cot \frac{1}{2} S \sin \frac{1}{2} c$, the vertices of all the triangles having the same base c, and the same surface S, will be situated on the little circle described from the point K as a pole, at a distance KC, such that $\cos KC = \sin PK \cos \frac{1}{2} c$.

For this beautiful theorem we are indebted to Lexell. (See Vol. V. Part I. of the *Nova Acta Petropolitana.*)

NOTE XI.

On Proposition 3, Book VII.

This Proposition may be more rigidly demonstrated by referring it to the Preliminary Lemmas in the following manner.



We shall first shew that the convex surface terminated by the edges AF, BG, and by the arcs A u B, F x G, cannot be less than the rectangle ABGF, the corresponding part of the inscribed prism's surface.

Let S be the surface in question; and, if possible, suppose the rectangle $AB \times AF = S + M$, M being a positive quantity.

Produce AF, the altitude of the prism and the cylinder, to a distance AF′, equal to n times AF, n being any whole number: if the prism and the cylinder are at the same time produced, the convex surface S′, included between the edges AF′, BG′, will evidently contain n times the surface F′ S; so that we shall have $S' = n S$; and because $n \times AF = AF'$, we shall have $AB \times AF = n S + n M = S' + n M$. Now n being any whole number taken at will, and M a given surface, n may be assumed, so that n M shall be greater than twice the segment A u B, since for this purpose it is enough to make $n > \frac{2 A u B}{M}$; hence in that case, the rectangle $AB \times AF'$, or the plane surface ABG′F′ must be greater than the enveloping surface, composed of the convex surface S′, and of two equal circular segments A u B, F′ x' G′. But, on the contrary, the latter surface is greater than the former, by the first Preliminary Lemma; hence in the first place, we cannot have $S < ABGF$.

We shall shew, in the second place, that the same convex surface S cannot be equal to that of the rectangle ABGF. For, if possible, suppose that by taking $AE = AB$, the convex surface AMK could be equal to the rectangle AFKE: through any point M in the arc AME, draw the chords AM, ME, and erect MN perpendicular to the plane of the base. The three rectangles AMNF, MEKN, AEKF, having the same altitude, are to each other as their bases AM, ME, AE. Now we have $AM + ME > AE$; hence the sum of the rectangles AMNF, MEKN is greater than the rectangle AFKE. The latter is, by hypothesis, equivalent to the convex surface AMK, composed of the two partial surfaces AN, MK. Hence the sum of the rectangles AMNF, MEKN is greater than the sum of the corresponding convex surfaces AN, MK. Hence one at least of the rectangles AMNF, MEKN must be greater than the corresponding convex surface; a consequence at variance with the first part demonstrated above. Hence, in the second place, the convex surface S cannot be equal to that of the corresponding rectangle ABGF.

It follows that we have $S > ABGF$, in other words, that the convex surface of the cylinder is greater than that of any inscribed prism.

By the very same mode of reasoning, we could prove the convex surface of the cylinder to be less than that of any circumscribed prism.

NOTE XII.

On the Equality and the Similarity of Polyedrons.

At the beginning of Euclid's Twelfth Book, we find definitions 9 and 10 conceived in these terms:

9. *Two solids are similar, when they are bounded by the same number of planes similar each to each.*

10. *Two solids are equal and similar, when they are bounded by the same number of planes equal and similar each to each.*

As the object of these definitions is one of the most difficult points in the Elements of Geometry, we shall examine it somewhat in detail, discussing at the same time the observations which Robert Simson has offered with regard to it, in his edition of Euclid, p. 388 *et seq.*

In the first place, we may remark with Simson, that definition 10 is not properly a definition, but in truth a theorem which needs to be demonstrated; for it is not evident, that two solids are equal from the circumstance alone that their faces are equal; and if this proposition is really correct, it ought to have been proved either by superposition, or in some other way. It afterwards appears, that Def. 9 is infected with the error of Def. 10. For if Def. 10 is not demonstrated, it is possible to imagine that two unequal and dissimilar solids having equal faces might exist; and in that case, by Def. 9, a third solid having its faces similar to those of the first two might be similar to each of them, and would thus be similar to two bodies of different forms; a conclusion which implies a contradiction, or at least does not agree with the meaning naturally attached to the word similar.

Several Propositions of Books XI. and XII. in Euclid are founded on Definitions 9 and 10; among others, Prop. 28. XI., upon which depends the measurement of prisms and pyramids. It appears, therefore, that Euclid's Elements are liable to the objection of containing a considerable number of propositions which are not rigorously demonstrated. There is one circumstance, however, tending to weaken this objection, which must not be omitted.

The figures whose equality or similarity Euclid demonstrates, by means of Definitions 9 and 10, are such that their solid angles do not combine more than three plane angles; and, if two solid angles are composed of three plane angles equal each to each, it is proved with sufficient clearness, in various parts of Euclid, that those solid angles are equal. On the other hand, if two polyedrons have their faces equal or similar each to each, the homologous solid angles will be composed of the same number of plane angles equal each to each. So long, therefore, as no more than three solid plane angles are joined in each solid angle, the homologous solid angles will evidently be equal. But, if the homologous faces are equal, and also the homologous solid angles, there can be no longer any doubt that the solids are equal; for they will coincide if applied to each other, or at least be symmetrical. The enunciation of Def. 9 and 10 appears therefore to be true and admissible, at least in the case of triple solid angles, the only case which Euclid attends to, and hence the reproach of inaccuracy directed against this author or his commentators, loses much of its weight, applying only to the want of some explanations and restrictions which he has not given.

We have still to examine if the enunciation of Def. 10, which is true in the case of triple solid angles, is true in general. Robert Simson maintains that it is not, and that two unequal solid angles may be formed, both of which shall be bounded by the same number of faces equal each to each. In support of this assertion, he brings forward an example, which may be generalized as follows.

If to any polyedron we add a pyramid, giving it one of the polyedron's faces for a base; if afterwards in place of adding, we cut off such a pyramid, by forming in the polyedron a cavity equal to it, we shall thus obtain two new solids, both having their faces equal each to each, yet themselves unequal.

There can be no doubt as to the inequality of the solids so constructed; but we may observe, that one of those solids contains re-entrant angles; and it is more than probable that Euclid meant to exclude all irregular bodies which have cavities or re-entrant angles, limiting himself to convex polyedrons. Admitting such a restriction, without which, moreover, various other propositions would not be true, the example adduced by Simson, concludes nothing against the definition or theorem of Euclid.

But however this may be, it results from all those observations that Euclid's 10th and 9th Definitions cannot be retained in their present form. Robert Simson cancels the definition of equal solids, which ought certainly to have its place among the theorems; and he defines *similar solids* as those which, being included under the same number of similar planes, have their solid angles equal each to each. This definition is no doubt true; but it has the inconvenience of containing several superfluous conditions. If the condition of the solid angles being equal were suppressed, we should again fall into the statement of Euclid, which is defective, because it presupposes the demonstration of a theorem concerning equal polyedrons. To avoid all embarrassment, we thought it best to divide the definition of similar triangular pyramids into two parts: first, we gave the definition of similar triangular pyramids, then we defined *similar solids* as those which have similar bases, and whose homologous vertices out of those bases are determined by triangular pyramids, similar each to each.

This definition requires two conditions for the bases, supposing them to be triangular; and three conditions for each vertex out of the base; so that, if S represent the number of solid angles in each of the two polyedrons, the similarity of these polyedrons will require $2+3\,(S-3)$ equal angles on both sides, or $3\,S-7$ conditions; and no one of these conditions is superfluous or contained in the others. For, in this definition, we regard two polyedrons as having simply the same number of vertices or solid angles; in which case, all the $3\,S-7$ conditions without one exception are required to make the two solids similar: but if we suppose them before-hand to be *of the same species*, in other words, to have the same number of faces, these faces compared with each other having also the same number of sides; then such a supposition, in the case where some faces have more than three sides, would include some conditions, and these would diminish the number $3\,S-7$ to the same extent; so that in place of $3\,S-7$ conditions, we should then require only $A-1$; (upon which see Note VIII). We now perceive what gives rise to the difficulty of settling a good definition of similar solids; it is that they may either be regarded as being of the same species, or only as having an equal number of solid angles. In the latter case, the whole difficulty vanishes; the $3\,S-7$ conditions implied in the definition must all be fulfilled before the solids can be similar; and from their similarity we conclude, *a fortiori*, that they are of the same species. As our definition was perfectly complete, we have deduced Simson's from it as a theorem.

It appears then, that in the Elements we might altogether omit the theorem concerning the equality of polyedrons; but as this is interesting of itself, the reader will not be displeased to find a demonstration of it here, and thus to have the theory of polyedrons complete in all respects.*

The question is, to know whether by varying the inclinations of the planes which compose the surface of a given convex polyedron, a second convex polyedron may be formed that shall be included under the same polygonal planes combined with each other in the same order.

We may observe, in the first place, that if there be a second polyedron satisfying the conditions of the problem, it cannot be the polyedron

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[*] The demonstration here given, with the exception of some developements, is the same as that which M. Cauchy lately communicated to the Institute. He has founded it on some ideas, which had been proposed for the same purpose, in the first edition of these Elements, page 327 *et seq.*

which is symmetrical to the given one, because in these two polyedrons the equal polyedrons are combined in an inverse order about the corresponding solid angles. Hence the consideration of symmetrical polyedrons must be entirely kept out of view in the object we are engaged with.

We may observe, in the second place, that if the given polyedron contains one or more triple solid angles, these angles are in their nature invariable, the knowledge of three plane angles being sufficient to determine the mutual inclination of their planes when joined into a solid angle. In the proposed solid, we are therefore at liberty to suppress all triangular pyramids which form the triple solid angles;* and if the new polyedron resulting from this suppression has still any triple solid angles, they two may be suppressed, and so on till we arrive at a polyedron every one of whose solid angles combines not less than four plane angles. In fact, if the proposed solid may alter its figure by any change in the inclinations of its planes, this alteration cannot take place in the triangular pyramids which are cut off; it must be wholly confined to the polyedron what remains after the suppression of all the triangular pyramids. In what follows, therefore, we shall only contemplate such polyedrons as have solid angles uniting at least four plane angles in each.

This being fixed, let S (see the next figure) be any solid angle in the polyedron; and from the vertex S as a centre, let a spherical surface be described whose intersections with the planes of the solid angle may form the spherical polygon ABCDEF. The sides of this polygon, AB, BC, &c. serve as measures to the plane angles ASB, BSC, &c., and are consequently invariable: as for the angles A, B, C, &c. of the polygon, each of them measures the inclination of two adjacent planes in the solid angle; thus the angle B measures the inclination of the planes ASB, BSC, which inclination, for the sake of brevity, we shall style the *inclination to the edge* SB; in like manner, the angle C measures the inclination to the edge SC; and so on.

We shall now be able to judge respecting the changes of figure in each solid angle S, by the changes of the spherical polygon ABCDEF, whose sides are constant, and whose angles may vary in any way provided the polygon do not cease to be convex. Now, in such polygons, the signs of variations in the angles present some remarkable laws, which we shall explain in the following Lemmas.

LEMMA I.

All the sides of a spherical polygon AB, BC, CD, DE, *with the exception of one* AF, *being given, if any one of the angles* B, C, D, E, *opposite the side* AF, *be made to vary, all the rest remaining constant, then will the side* AF *augment as the angle augments, and diminish as*

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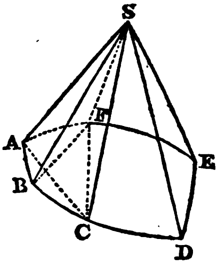
* If the same edge were common to two triple solid angles, we could only, in the first operation, suppress one of these angles.

it diminishes. In all cases the polygon is supposed to be convex both before and after its change of figure.



Suppose, first, that the angle B is made to vary, the other three C, D, E remaining constant; if BF is joined, the figure BCDEF will undergo no change, and BF will be constant. Hence we shall have a spherical triangle ABF, whose sides AB, BF are constant, while its angle ABF varies by the same quantity as the angle ABC of the polygon, since the part FBC remains constant. Now, by properties which are admitted,* we know that the side AF will increase if the angle ABF increases, and diminish if the angle ABF diminishes.

Next, suppose the angle C varies, the other three B, D, E being constant; if the diagonals AC, FC are drawn, these diagonals will evidently remain constant, and likewise the angles ACB, FCD: hence we shall again have a spherical triangle ACF, whose sides AC, CF are constant, while its angle ACF varies by the same quantity as the angle C of the polygon; whence we shall, in like manner, infer that the side AF will augment if the angle C augments, and diminish if that angle diminishes.

The same reasoning would evidently apply to the variation of either of the angles D and E; it would likewise evidently hold good for any other spherical polygon having more than three sides. Hence, in all cases, our conclusion will agree with the enunciation of the Proposition, still however supposing, that before as well as after its change of figure, the polygon is convex. This restriction is necessary; for if the angle E, for example, should diminish till the point F fell on the diagonal AE, then AF would be a *minimum;* and if reckoning from this point, the angle E continued to diminish, the side AF would evidently augment instead of diminishing; but in this latter case, the angle AFE would become re-entrant, and the polygon would cease to be convex.

Cor. The same things being admitted, if several of the angles opposite the undetermined side AF augment, and none of them diminish, the side AF will of necessity augment by means of all the joint variations. The contrary effect will take place, if several of the angles opposite the side AF diminish and none of them augment.

For, if by reason of the simultaneous increase or diminution, the angles A, B, C, &c., of the polygon are to be changed into A′, B′, C′, &c., we may pass successively from the proposed polygon to that which contains only one varied angle A′; from this latter, to the polygon which contains only two varied angles A′ and B′; and so on. Now, in each of these transitions, the application of the Theorem above demonstrated is manifest, and always leads to the same conclusion.

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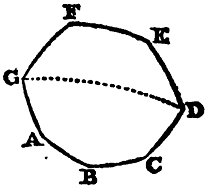
* This proposition is demonstrated in the same manner as Prop. 10. Book I. for rectilineal triangles.

LEMMA II.

Given a spherical polygon having more than three sides, and all of them constant, if the angles are made to vary in any way, so the polygon do not cease to be convex; if, farther, the sign + is put at the vertex of each angle that augments, the sign — at the vertex of each angle that diminishes, no sign whatever being put at the angles that remain constant; then I assert that, in going quite round the polygon, we shall find at least four changes of sign from one vertex to the next vertex.

First. If n is the number of angles in the polygon, there cannot be $n-2$ consecutive angles, all of which increase at once, or of which some increase and the others remain constant; for if either of these results took place, it would follow by the Corollary of the preceding Lemma, that the side of the polygon, lying opposite to those $n-2$ angles, would increase; which contradicts the hypothesis that all the sides are constant. For a like reason, there cannot be $n-2$ consecutive angles, all of which diminish at once, or of which some diminish, the others remaining constant. Hence, in the series of $n-2$ consecutive angles, there must occur at least one change of sign; and *a fortiori,* must this change be observed in the series of n consecutive angles, in going quite round the polygon.

Secondly. The variations in the angles of the polygon cannot be such as to offer only one series of + signs, and one series of — signs, thus offering only two changes of sign in the whole circuit of the polygon.



For, let A, B, C be the three angles marked with the sign +, and D, E, F, G the angles, marked with the sign —; an hypothesis which includes all such as refer to a less number of signs in each series, when some of the angles are invariable. If the figure represents the initial state of the polygon, the diagonal GD must increase when the three angles A, B, C, or only some of them increase; but the same diagonal GD, as belonging to the polygon GFED, of which the other sides are constant, must diminish along with the angles F and E, or at least remain constant, if of the four angles D, E, F, G, there be only D and G, or merely one of them which diminishes: hence the hypothesis in question cannot be correct; hence the variation of the angles cannot be such as to offer only two series, one of signs +, another of signs —.

Thirdly. It is farther impossible that in going round the polygon we should not find three alternate series of signs +, and of signs —; for on this hypothesis the first and the third series would be of the same sign, and following each other immediately, would form only a single series; from which it appears, that in going round the polygon there would in reality be only two series, the one of signs +, the other of signs —; which we have already shewn to be impossible.

Hence, finally, the changes of sign to be found in going quite round the polygon will at least amount to four.

Cor. What we have just proved with regard to spherical polygons is immediately applicable to the solid angles, of which these polygons are the measure. Thus *in a convex solid, an angle being given which unites more than three plane angles, if the inclinations to the edges are made to vary in any way, provided the solid angle do not cease to be convex; if farther the sign + or the sign — is put on each edge according as the inclination to it augments or diminishes, no sign whatever being marked upon the edges whereof the inclination continues constant, I assert that in going quite round the solid angle, we shall find at least four changes of sign from one edge to the next.*

By means of this Proposition, and of Euler's theorem concerning polyedrons (Prop. 25. VII.) we are now in a condition to demonstrate the following theorem under its most general form.

THEOREM.

Given a convex polyedron, each of whose solid angles unites more than three plane angles, it is impossible to vary the inclinations of the planes in this solid, so as to produce a second polyedron formed with the same planes arranged with each other in the same order as in the given polyedron.

In demonstrating this Proposition, two cases must be distinguished, according as the inclination to all the edges is made to vary, or only to some of them.

First Case.

Suppose the inclinations on all the edges are made to vary at once; and let N be the total number of changes of sign that can be found from one edge to another, in going round each solid angle.

In Lemma II. we saw, that for each solid angle, the number of changes of sign cannot be less than four.

Hence if we call S the number of solid angles, we shall have $N \geqslant 4S$, the sign $\geqslant$ not excluding equality.

Now it is plain that two consecutive edges of a solid angle always belong to one face in a polyedron, and belong only to one; hence the total number of changes of sign observed in the consecutive edges of each solid angle, must be equal to the total number of changes of sign observed in the consecutive sides of each face; for there is no change of sign in the one system, which does not correspond to a similar change in the other.

Now for each triangular face, the number of changes of sign cannot be greater than two; since by making the series + — + or the series — + — return into itself, we obtain only two changes of sign.

For each quadrangular face, the number of changes of sign is evidently four at most.

In general, if the number of sides belonging to a face is even $= 2n$, the greatest number of changes of sign that can be found in going round the sides is $2n$, which will occur when the sides have the signs + and — alternately.

But if the number of sides belonging to a face is odd $= 2n+1$, the greatest number of changes of sign will be $2n$ only; because by giving the signs + and — alternately to the sides, the first and the last will necessarily have the same sign, which makes one change fewer than the face has sides.

This being proved, let a represent the number of triangles, b the number of quadrilaterals, c the number of pentagons, &c, which compose the surface of the given polyedron; it follows, from what we have said above, that the total number of changes of sign to be found in going round each face, will not exceed $2a$ for triangular, $4b$ for quadrilateral faces, $4c$ for those of five sides, $6d$ for those of six. Hence we shall have

$$N \leq 2a + 4b + 4c + 6d + 6e + 8f + 8g + \&c.$$

Let A be the number of the polyedron's edges, H that of its faces; we shall have

$$2A = 3a + 4b + 5c + 6d + 7e + 8f + 9g + \&c.$$
$$H = a + b + c + d + e + f + g + \&c.$$

But according to Euler's theorem, $S + H = A + 2$; hence $4S = 8 + 4A - 4H$, and substituting,

$$4S = 8 + 2a + 4b + 6c + 8d + 10e + \&c,$$

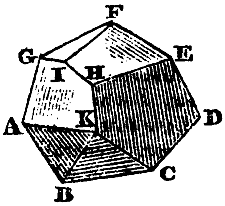
Compare this value to the limit found above; we find

$$N \leq 4S - 8$$

But we cannot at once have $N \geq 4S$, and $N \leq 4S - 8$; hence it is impossible that the inclinations to the edges of the polyedron can vary all at once, without destroying the coherence of the planes which form the surface of the polyedron.

Second Case.

Suppose now that the inclinations to the edges do not all vary at once; that there are some which remain constant.



Let FI be one of these edges; we might imagine it to be suppressed, and the two adjacent faces FIG, EFIH combined into a single unplane face, terminated by the contour EFGIH whose form is invariable. Let us designate by S′, H′ and A′, what the numbers S, H, and A become after the suppression of an edge; we shall have $H' = H - 1$, and $A' = A - 1$: also we have $S' = S$, the number of solid angles being the same in both solids; hence we shall have $S' + H' - A' = S + H - A = 2$. Whence it is evident that Euler's theorem holds good likewise in the new solid, which contains one edge less, and one face less, two faces being combined into a single unplane face.

If from this second solid, there is cut off another of the edges on which the inclination remains invariable, the suppression of this edge will again occasion the reunion of two contiguous faces into one; and we shall be able to shew, as before, that Euler's theorem holds good in the third solid which results from the suppression of two edges.

We might continue to suppress as many edges as we please, provided this suppression carried with it the suppression of no solid angle; and Euler's theorem would always hold good in the remaining solid. This truth, indeed, may be seen directly and generally, from examining the demonstration we have already given of Euler's theorem: which demonstration, in fact, does not suppose that the faces of the polyedron are plane; but would equally continue valid, though these faces were terminated by contours not situated in the same planes; it only supposes, that, as in our construction, each contour is represented by a spherical polygon, and that the sum of all the surfaces of these polygons is equal to the surface of the sphere. Nor is it even necessary that these polygons be all convex; it is enough that each of them may be regarded as the sum of several convex polygons; a condition which will always be observed, when, even by the suppression of several edges belonging to the given polyedron, several plane faces are combined into a single unplane face; because, in that case, the spherical polygon representing the latter face, will be composed of the sum of the convex spherical polygons which represented the suppressed plane faces.

Let us now proceed to the case where the suppression of the edges on which the inclination does not vary, carries with it the suppression of one or several solid angles, either because the inclinations to all the edges in each of these angles are invariable, or because these inclinations being capable of varying only on three edges, would, in that case, be of necessity invariable.

Suppose, first, that only one solid angle is suppressed; and let m be the number of faces in this angle, or the number of edges ending in its vertex. By suppressing the solid angle in question, we shall, at the same time, suppress m edges, and the m faces forming the solid angle will be reduced to a single face; hence, designating by S′, A′, H′, what the numbers S, A, H become after the suppression of a solid angle, we shall have $S' = S - 1$, $A = A' - m$, $H' = H - (m - 1)$. From this we obtain $S' + H' - A' = S + H - A = 2$; hence Euler's theorem still holds good in this new solid.

It is now evident, that in the given polyedron, as many solid angles as we please may be suppressed, and that Euler's theorem will always continue true with regard to the remaining polyedron; for by suppressing the solid angles one by one, we successively obtain different polyedrons, of which two consecutive polyedrons always fall under the case we have just examined.

Hence, generally, if in the proposed polyedron, all those edges are suppressed on which the inclination does not vary; whether by this suppression the number of solid angles remain the same, or become less, the remaining polyedron will always satisfy Euler's theorem; in other words, naming s, h, a, in this polyedron the quantities which correspond to S, H, A, in the proposed polyedron, we shall constantly have $s + h - a = S + H - A = 2$.

But in this last solid, the inclinations on the edges must all vary at once, all the edges on which the inclination does not vary being suppressed; hence this solid comes back under the first case; hence the simultaneous change of all the inclinations cannot take place without destroying the solid.

Hence, finally, no convex polyedron whatever can be changed into another convex polyedron, included under the same polygonal planes, arranged in the same order with regard to each other.

END OF THE NOTES.

TREATISE

ON

TRIGONOMETRY.

TRIGONOMETRY has for its object the solution of triangles, that is, the determination of their sides and angles, when a sufficient number of those sides and angles is given.

In rectilineal triangles, it is sufficient to know three of the six parts which compose them, provided there be a side among these three. If the three angles only were given, it is obvious that all similar triangles would answer the question.

In spherical triangles, any three given parts, angles or sides, are always sufficient to determine the triangle; because, in triangles of this sort, the absolute magnitude of the sides is not considered, but only their relation to the quadrant, or the number of degrees which they contain.

In the Problems annexed to Book II., we have already seen how rectilineal triangles are constructed by means of three given parts. Propositions 24 and 25 of Book V. give likewise an idea of the constructions, by which the analogous cases of spherical triangles might be resolved. But those constructions, though perfectly correct in theory, would give only a moderate approximation in practice,* on account of the imperfection of the instruments required in constructing them: they are called *graphic methods*. Trigonometrical methods, on the contrary, being independent of all mechanical operations, give solutions with the utmost accuracy: they are founded upon the properties of lines called *sines*, *cosines*, *tangents*, &c. which furnish a very simple mode of expressing the relations that subsist between the sides and angles of triangles.

We shall first explain the properties of those lines, and the principal formulas derived from them; formulas which are of great use in all the branches of mathematics, and which even furnish means of improvement to algebraical analysis. We shall next apply those results to the solution of rectilineal triangles, and then to that of spherical triangles.

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* We are naturally required to distinguish the figures which serve only to direct our reasoning in the demonstration of a theorem or the solution of a problem, from the figures which are constructed to find some of their dimensions. The first are always supposed to be exact; the second, if not exactly drawn, will give false results.

DIVISION OF THE CIRCUMFERENCE.

I. Till very recent times, geometers were agreed in dividing the circumference into 360 equal parts, called *degrees*, the degree into 60 *minutes*, the minute into 60 *seconds*, and so on. This method seemed to be of advantage in practice, on account of the great number of divisors which 60 and 360 have; but it was, in reality, subject to the inconvenience of complex numbers; and it frequently impeded the rapidity of calculation.

The mathematicians, to whom we are indebted for the new system of weights and measures, conceived that it would be of great advantage to introduce the decimal division in the measurement of angles. Upon this principle, considering as their fundamental unit the quarter of the circumference, or the quadrant, which measures the right angle, they have divided this unit into 100 equal parts, called *degrees*, the degree into 100 *minutes*, and the minute into 100 *seconds*.

We shall henceforth exclusively employ the new or decimal division of the circumference.* It is most agreeable to the nature of our arithmetic, and best fitted to abridge calculations.

II. Degrees, minutes, and seconds, are respectively designated by these characters: °, ′, ″: thus the expression 16° 6′ 75″ represents an arc, or an angle, of 16 degrees 6 minutes 75 seconds. The relation of this same arc to the quadrant, taken for unit, would be expressed by 0. 160675. It likewise appears, that the angle, measured by this arc, is to the right angle as 160675 to 1000000, a ratio which could not have been so easily deduced from the expressions furnished by the old division of the circumference.

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* It is much to be regretted, that this centesimal division of the quadrant had not been proposed at an earlier period in the history of science. The advantages it has over the common or *nonagesimal* system are numerous and palpable; but as all our trigonometrical calculations have for so many centuries been performed, and their numerous results expressed, with a constant reference to the latter, the nonagesimal system has at length become as it were a part of the vernacular language of mathematicians; and bids fair to be as indestructible as a vernacular language. At all events, the French are as yet solitary in their rejection of it; and the general adoption of their new system, if it ever take place, which appears extremely problematical, must certainly be a very distant event. For these reasons, we could have wished that the nonagesimal scale had been followed in the present treatise; or that, without altering any material part of the work, we could have substituted it in our translation. This was found to be impracticable; but the present arrangement, while it exhibits the advantages of the new system, followed in all the most distinguished mathematical works of a recent date, will in other respects occasion very little inconvenience. The reader has only to bear in mind, that 200°, 100°, 50°, &c., when mentioned in the text, mean respectively 180°, 90°, 45°, &c., in the common tables; and that, generally, any angle may be reduced from degrees in the centesimal scale to degrees in the nonagesimal, by diminishing it in the ratio of 400 to 360, or of 10 to 9. Wherever this process could be attended with the slightest difficulty, or the omission of it might create any ambiguity, we shall subjoin a note, in which the required reduction will be exhibited.—Ed.

Arcs and angles are, in calculation, expressed indifferently by numbers of degress, minutes, and seconds. Thus we shall designate the right angle, or the quadrant, by 100°; two right angles, or the semi-circumference, by 200°; four right angles, or the whole circumference, by 400°; and so on.

III. The *complement* of an angle, or of an arc, is what remains after taking that angle or that arc from 100°. Thus an angle of 25° 40′, has for its complement 74° 60′; an angle of 12° 4′ 62″, has for its complement 87° 95′ 38″.

In general, A being any angle or any arc, 100°—A is the complement of that angle or arc. Whence it is evident that, if the angle or arc is greater than 100°, its complement will be negative. Thus the complement of 160° 84′ 10″ is — 60° 84′ 10″. In this case, the complement, positively taken, would be the quantity requiring to be subtracted from the given angle or arc, that the remainder might be equal to 100°.

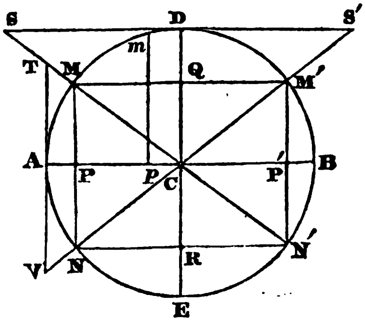
The two angles of a right-angled triangle, are, together, equal to a right-angle: they are, therefore, complements of each other.

IV. The *supplement* of an angle, or of an arc, is what remains after taking that angle or arc from 200°, the value of two right angles, or of a semi-circumference. Thus A being any angle or arc, 200° — A is its supplement.

In any triangle, an angle is the supplement of the sum of the two others, since the three together make 200°.

The angles of triangles rectilineal and spherical, and the sides of the latter, have their supplements always positive; for they are always less than 200°.

GENERAL IDEAS RELATING TO SINES, COSINES, TANGENTS, &c.



V. The *sine* of the arc A M, or of the angle ACM, is the perpendicular MP let fall from one extremity of the arc, on the diameter which passes through the other extremity.

If at the extremity of the radius CA, the perpendicular AT is drawn to meet the production of the radius CM, the line AT, thus terminated, is called the *tangent*, and CT the *secant* of the arc AM, or of the angle ACM.

These three lines MP, AT, CT, dependent upon the arc AM, and always determined by it and the radius, are thus designated: MP=*sin* AM, or *sin* ACM, AT=*tang* AM, or *tang* ACM, CT=*sec* AM, or *sec* ACM.

VI. Having taken the arc AD equal to a quadrant, from the points M and D draw the lines MQ, DS perpendicular to the radius CD, the one terminated by that radius, the other terminated by the radius CM produced; the lines MQ, DS and CS, will, in like manner, be the sine, tangent, and secant of the arc MD, the complement of AM. For the sake of brevity, they are called the *cosine*, *cotangent*, and *cosecant*, of the arc AM, and are thus designated: MQ=*cos* AM, or *cos* ACM, DS=*cot* AM, or *cot* ACM, CS=*cosec* AM or *cosec* ACM. In general, A being any arc or angle, we have *cos* A=*sin* (100°—A), cot A=*tang* (100°—A), *cosec* A=*sec* (100°—A).

The triangle MQC is, by construction, equal to the triangle CPM; consequently CP=MQ: hence in the right-angled triangle CMP, whose hypotenuse is equal to the radius, the two sides MP, CP are the sine and cosine of the arc AM. As to the triangles CAT, CDS, they are similar to the equal triangles CPM, CQM; hence they are similar to each other. From these principles, we shall very soon deduce the different relations which exist between the lines now defined: before doing so, however, we must examine the progressive march of those lines, when the arc to which they relate increases from zero to 200°.

VII. Suppose one extremity of the arc remains fixed in A, while the other extremity, marked M, runs successively throughout the whole extent of the semicircumference, from A to B in the direction ADB.

When the point M is at A, or when the arc AM is zero, the three points T, M, P, are confounded with the point A; whence it appears that the sine and tangent of an arc zero are zero, and the cosine and secant of this same arc, are each equal to the radius. Hence if R represent the radius of the circle, we have

sin o=o, *tang* o=o, *cos* o=R, *sec* o=R.

VIII. As the point M advances towards D, the sine increases, and likewise the tangent and the secant; but the cosine, the cotangent, and the cosecant, diminish.

When the point M is at the middle of AD, or when the arc AM is 50°, and also its complement MD, the sine MP is equal to the cosine MQ or CP; and the triangle CMP, now become isosceles, gives the proportion MP : CM : : 1 : $\sqrt{2}$, or *sin* 50° : R : : 1 : $\sqrt{2}$. Hence *sin* 50° = *cos* 50° = $\frac{R}{\sqrt{2}}$ = $\frac{1}{2}$R $\sqrt{2}$. In this same case, the triangle CAT becomes isosceles and equal to the triangle CDS; whence, the tangent of 50° and its cotangent, are each equal to the radius, and consequently we have *tang* 50° = *cot* 50° = R.

IX. The arc AM continuing to increase, the sine increases till M arrives at D; at which point the sine is equal to the radius, and the cosine is zero. Hence we have *sin* 100° = R, *cos* 100° = o; and it may be observed, that these values are a consequence of the values already found for the sine and cosine of the arc zero; because, the complement of 100° being zero, we have *sin* 100° = *cos* o° = R, and *cos* 100° = *sin* o° = o.

As to the tangent, it increases very rapidly as the point M approaches D; and finally when this point reaches D, the tangent properly exists no longer, because the lines AT, CD, being parallel, cannot meet. This is expressed by saying that the tangent of 100° is infinite; and we write *tang* 100° = ∞.

The complement of 100° being zero, we have *tang* o = *cot* 100°, and *cot* o = *tang* 100°. Hence *cot* o = ∞, and cot 100° = o.

X. The point M continuing to advance from D towards B, the sines diminish and the cosines increase. Thus M′P′ is the sine of the arc AM′, and M′Q or CP′ its cosine. But the arc M′B is the supplement of AM′, since AM′+M′B is equal to a semicircumference; besides, if M′M is drawn parallel to AB, the arcs AM, BM′, which are included between parallels, will evidently be equal, and likewise the perpendiculars or sines MP, M′P′. Hence, *the sine of an arc or of an angle is equal to the sine of the supplement of that arc or angle.*

The arc or angle A has for its supplement 200°—A: hence generally, we have

sin A = (*sin* 200°—A.)

The same property might also be expressed by the equation *sin* (100°+B) = sin (100°—B), B being the arc DM or its equal DM′.

XI. The same arcs AM′, AM which are supplements of each other, and which have equal sines, have also equal cosines CP′, CP; but it must be observed, that these cosines lie in different directions. This difference of situation is expressed in calculation by a difference in the signs; so that if the cosines of arcs less than 100° are considered as *positive* or affected with the sign +, the cosines of arcs greater than 100° must be considered as *negative* or affected with the sign $-$. Hence, generally, we shall have

$$\cos A = -\cos(200-A^\circ)$$

or $\cos(100^\circ+B) = -\cos(100^\circ-B)$; that is, *the cosine of an arc or of an angle greater than* 100° *is equal to the cosine of its supplement negatively taken.*

The complement of an arc greater than 100° being negative (Art. 3), it is natural that the sign of that complement should be negative: but to render this truth still more palpable, let us seek the expression of the distance from the point A to the perpendicular MP. Making the arc $AM = x$, we have $CP = \cos x$, and the required distance $AP = R - \cos x$. The same formula must express the distance from the point A to the straight line MP, whatever be the magnitude of the arc AM originating in the point A. Suppose then that the point M come to M′, so that x designates the arc AM′; we have still at this point, $AP' = R - \cos x$: hence $\cos x = R - AP' = AC - AP = -CP$; which shews that $\cos x$ is negative in that case: and because $CP' = CP = \cos(200^\circ - x)$, we have $\cos x = -\cos(200^\circ - x)$, as we found above.

From this it appears, that an obtuse angle has the same sine and the same cosine as the acute angle which forms its supplement; only with this difference, that the cosine of the obtuse angle must be affected with the sign $-$. Thus we have $\sin 150^\circ = \sin 50^\circ = \frac{1}{2}R\sqrt{2}$, and $\cos 150^\circ = -\cos 50^\circ = -\frac{1}{2}R\sqrt{2}$.

As to the arc ADB, which is equal to the semicircumference, its sine is zero, and its cosine is equal to the radius taken negatively: hence we have $\sin 200^\circ = 0$, and $\cos 200^\circ = -R$. This might also be derived from the formulas $\sin A = \sin(200^\circ - A)$, and $\cos A = -\cos(200^\circ - A)$, by making $A = 200^\circ$.

XII. Let us now examine what is the tangent of an arc AM′ greater than 100°. According to the Definition, this tangent is determined by the concourse of the lines AT, CM. These lines do not meet in the direction AT; but they meet in the opposite direction AV; whence it is obvious that the tangent of an arc greater than 100° must be negative. Also, because AV is the tangent of the arc AN, the supplement of AM′ (since NAM is a semicircumference), it follows that *the tangent of an arc or of an angle greater than* 100° *is equal to that of its supplement, taken negatively;* so that we have $\tang A = -\tang(200-A)$.

The same thing is true of the cotangent represented by DS′, which is equal to DS the cotangent of AM, and in a different direction. Hence we have likewise $\cot A = -\cot(200^\circ - A)$.

The tangents and cotangents are therefore negative, like the cosines, from 100° to 200°. And at this latter limit, we have $tang\ 200^\circ = o$ and $cot\ 200^\circ = -cot\ o = -\infty$.

* XIII. In trigonometry, the sines, cosines, &c. of arcs or angles greater than 200° do not require to be considered; the angles of triangles, rectilineal as well as spherical, and the sides of the latter being always comprehended between 0 and 200°. But in various applications of geometry, there is frequently occasion to reason about arcs greater than the semicircumference, and even about arcs containing several circumferences. It will therefore be necessary to find the expression of the sines and cosines of those arcs whatever be their magnitude.

We observe, in the first place, that two equal arcs AM, AN with contrary signs, have equal sines MP, PN with contrary signs; while the cosine CP is the same for both. Hence we have in general

$$sin\ (-x) = -sin\ x$$
$$cos\ (-x) = cos\ x,$$

formulas which will serve to express the sines and cosines of negative arcs.

From 0° to 200° the sines are always positive, because they always lie on the same side of the diameter AB; from 200° to 400°, the sines are negative, because they lie on the opposite side of their diameter. Suppose $ABN' = x$ an arc greater than 200°; its sine $P'N'$ is equal to PM, the sine of the arc $AM = x - 200^\circ$. Hence we have in general

$$sin\ x = -sin\ (x - 200^\circ)$$

This formula will give us the sines between 200° and 400°, by means of the sines between 0° and 200°: in particular it gives $sin\ 400^\circ = -sin\ 200^\circ = 0$; and accordingly, if an arc is equal to the whole circumference, its two extremities will evidently be confounded together at the same point, and the sine be reduced to zero.

It is no less evident, that if one or several circumferences were added to any arc AM, it would still terminate exactly at the point M, and the arc thus increased would have the same sine as the arc AM; hence if C represent a whole circumference or 400°, we shall have

$$sin\ x = sin\ (C + x) = sin\ (2C + x) = sin\ (3C + x), \text{ \&c.}$$

The same observation is applicable to the cosine, tangent, &c.

Hence it appears, that whatever be the magnitude of x the proposed arc, its sine may always be expressed, with a proper sign, by the sine of an arc less than 100°. For, in the first place, we may subtract 400° from the arc x as often as they are contained in it; and y being the remainder, we shall have $\sin x = \sin y$. Then if y is greater than 200°, make $y = 200 + z$, and we have $\sin y = -\sin z$. Thus all the cases are reduced to that in which the proposed arc is less than 200°; and since we farther have $\sin (100 + x) = \sin (100 - x)$, they are likewise ultimately reducible to the case, in which the proposed arc is between zero and 100°.

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* The sections marked thus (*) are either of greater difficulty, or of less extensive use. They may be omitted without violating the continuity of the reasoning.—Ed.

*XIV. The cosines are always reducible to sines, by means of the formula $\cos A = \sin (100° - A)$; or if we require it, by means of the formula $\cos A = \sin (100° + A)$: and thus, if we can find the value of the sines in all possible cases, we can also find that of the cosines. Besides, the figure will easily shew us that the negative cosines are separated from the positive cosines by the diameter DE; all the arcs whose extremities fall on the left side of DE, having a positive cosine, while those whose extremities fall on the right have a negative cosine.

Thus from 0° to 100° the cosines are positive; from 100° to 300° they are negative; from 300° to 400° they again become positive; and after a whole revolution, they assume the same values as in the preceding revolution, for $\cos (400° + x) = \cos x$.

From these explanations, it will evidently appear, that the sines and cosines of the various arcs which are multiples of the quadrant, have the following values:

$\sin 0° = 0$	$\sin 100° = R$	$\cos 0° = R$	$\cos 100° = 0$
$\sin 200° = 0$	$\sin 300° = -R$	$\cos 200° = -R$	$\cos 300° = 0$
$\sin 400° = 0$	$\sin 500° = R$	$\cos 400° = R$	$\cos 500° = 0$
$\sin 600° = 0$	$\sin 700° = -R$	$\cos 600° = -R$	$\cos 700° = 0$
$\sin 800° = 0$	$\sin 900° = R$	$\cos 800° = R$	$\cos 900° = 0$
&c.	&c.	&c.	&c.

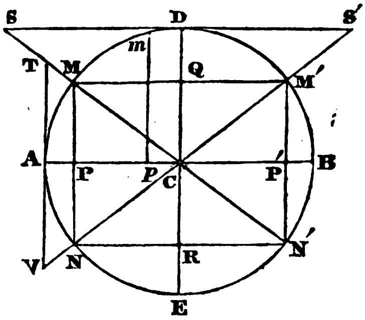
And generally, k designating any whole number, we shall have

$$\begin{aligned} &\sin 2k.100° = 0, && \cos (2k+1).100° = 0, \\ &\sin (4k+1).100° = R, && \cos 4k.100° = R, \\ &\sin (k-1).100° = -R, && \cos (4k+2).100° = -R \end{aligned}$$

What we have just said concerning the sines and cosines renders it unnecessary for us to enter into any particular detail respecting the tangents, cotangents, &c. of arcs greater than 200°: the values of these quantities are always easily deduced from those of the sines and cosines of the same arcs; as we shall see by the formulas, which we now proceed to explain.

THEOREMS AND FORMULAS RELATING TO SINES, COSINES, TANGENTS, &c.

XV. *The sine of an arc is half the chord which subtends a double arc.*



For the radius CA, perpendicular to the chord MN, bisects this chord, and likewise the arc MAN; hence MP, the sine of the arc MA, is half the chord MN which subtends the arc MAN, the double of MA.

The chord which subtends the sixth part of the circumference is equal to the radius; hence $\sin \frac{400^\circ}{12}$ or $\sin 33^\circ\frac{1}{3} = \frac{1}{2}R$, in other words, the sine of a third part of the right angle is equal to the radius.

XVI. *The square of the sine of an arc, together with the square of the cosine, is equal to the square of the radius; so that in general terms we have* $\sin{}^2A + \cos{}^2A = R^2$[*]

This property results immediately from the right-angled triangle CMP, in which $MP^2 + CP^2 = CM^2$.

It follows that when the sine of an arc is given, its cosine may be found, and *vice versa*, by means of the formulas $\cos A = \pm\sqrt{(R^2 - \sin{}^2A)}$, and $\sin A = \pm\sqrt{(R^2 - \cos{}^2A)}$. The sign of these formulæ is ambiguous, because the same sine MP answers to the two arcs AM, AM', whose cosines CP, CP' are equal and have contrary signs; as the same cosine CP answers to the two arcs AM, AN, whose sines MP, PN are also equal, and have contrary signs.

Thus, for example, having found $\sin 33^\circ\frac{1}{3} = \frac{1}{2}R$, we may deduce from it $\cos 33^\circ\frac{1}{3}$ or $\sin 66^\circ\frac{2}{3} = \sqrt{(R^2 - \frac{1}{4}R^2)} = \sqrt{\frac{3}{4}R^2} = \frac{1}{2}R\sqrt{3}$.

XVII. *The sine and cosine of the arc* A *being given, the*

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[*] By *sin* ^2A is here meant the square of *sin* A; and, in like manner, by *cos* ^2A is meant the square of *cos* A.

$CB : CI :: CE : CK$, or $R : \cos b :: \cos a : CK = \frac{\cos a \cos b}{R}$.

The triangles DIL, CBE, having their sides, are perpendicular each to each, are similar, and give the proportions,

$CB : DI :: CE : DL$, or $R : \sin b :: \cos a : DL = \frac{\cos a \cos b}{R}$.

$CB : DI :: BE : IL$, or $R : \sin b :: \sin a : IL = \frac{\sin a \sin b}{R}$.

But we have

$IK + DL = DF = \sin(a+b)$, and $CK - IL = CF = \cos(a+b)$.

Hence

$$\sin(a+b) = \frac{\sin a \cos b + \sin b \cos a}{R}$$

$$\cos(a+b) = \frac{\cos a \cos b - \sin a \sin b}{R}.$$

The values of $\sin(a-b)$ and of $\cos(a-b)$ might be easily deduced from these two formulas; but they may be found directly by the same figure. For, produce the sine DI till it meets the circumference at M; then we have $BM = BD = b$, and $MI = ID = \sin b$. Through the point M, draw MP perpendicular and MN parallel to AC: since $MI = DI$, we have $MN = IL$, and $IN = DL$. But we have $IK - IN = MP = \sin(a-b)$, and $CK + MN = CP = \cos(a-b)$; hence

$$\sin(a-b) = \frac{\sin a \cos b - \sin b \cos a}{R}$$

$$\cos(a-b) = \frac{\cos a \cos b + \sin a \sin b}{R}.$$

These are the formulas which it was required to demonstrate.

* The preceding demonstration may seem defective in point of generality, since, in the figure which we have followed, the arcs a and b, and even $a+b$, are supposed to be less than 100°. But first the demonstration is easily extended to the case in which a and b being less than 100°, their sum $a+b$ is greater than 100°. Then the point F would fall on the production of AC, and the only change required in the demonstration should be that of taking $\cos(a+a) = -CF$; but as we should, at the same time, have $CF = IL - CK$, it would still follow that $\cos(a+b) = CK - IL$ or $R \cos(a+b) = \cos a \cos b - \sin a \sin b$.

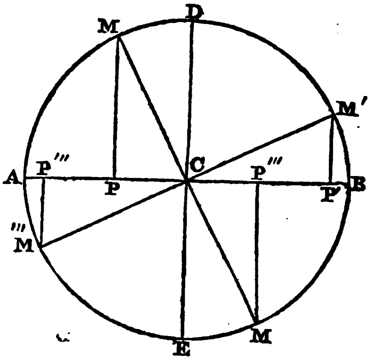
Now suppose the formulas

$$R \sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$R \cos(a+b) = \cos a \cos b - \sin a \sin b$$

to be acknowledged as correct for all the values of a and b less than the limits A and B; then will they also be true when these limits are $100^\circ+\mathrm{A}$ and B.

For, in general, whatever be the arc x, we have

$$\sin(100^\circ+x)=\cos x$$
$$\cos(100^\circ+x)=-\sin x.$$



These equations are evidently accurate when $x\angle 100^\circ$; and we easily discover their correctness whatever be the value of x, by inspecting this figure, in which MM′ and M′M‴ are two diameters perpendicular to each other; and in which we may substitute for x the values AM, ADM, ADBM″, ADBEM‴, or these values increased by as many circumferences as we please.

This being granted, put $x=m+b$; we have

$$\sin(100^\circ+m+b)=\cos(m+b)$$
$$\cos(100^\circ+m+b)=-\sin(m+b).$$

But, by hypothesis, the values of the second members are known, so long as m and b do not exceed the limits A and B; hence, according to this same hypothesis, we have

$$\mathrm{R}\sin(100^\circ+m+b)=\cos m\cos b-\sin m\sin b$$
$$\mathrm{R}\cos(100^\circ+m+b)=-\sin m\cos b-\cos m\sin b$$

Put $100^\circ+m=a$; since $\sin(100^\circ+m)=\cos m$ and $\cos(100^\circ+m)=-\sin m$, it follows that $\cos m=\sin a$ and $\sin m=-\cos a$; hence, by substituting this in the preceding equations, we have

$$\mathrm{R}\sin(a+b)=\sin a\cos b+\cos a\sin b$$
$$\mathrm{R}\cos(a+b)=\cos a\cos b-\sin a\sin b.$$

From which it appears that these formulas, at first proved only within the limits $a\angle\mathrm{A}$, $b\angle\mathrm{B}$, are now proved within the more extensive limits $a\angle 100^\circ+\mathrm{A}$, $b\angle\mathrm{B}$. But, in the very same way, the limit of b might be carried 100° farther; then so also might that of a, and the process might be continued indefinitely; hence the formulas in question hold good whatever be the magnitude of the arcs a and b.

Since the arc a is formed from the sum of the two arcs $a-b$ and b, by the preceding formulas we shall have

$$R \sin a = \sin (a-b) \cos b + \cos (a-b) \sin b,$$
$$R \cos a = \cos (a-b) \cos b - \sin (a-b) \sin b.$$

And from these we find

$$R \sin (a-b) = \sin a \cos b - \sin b \cos a$$
$$R \cos (a-b) = \cos a \cos b + \sin a \sin b.$$

XX. If, in the formulas of the preceding Article, we make $b=a$, the first and the third will give

$$\sin 2a = \frac{2 \sin a \cos a}{R}, \quad \cos 2a = \frac{\cos^2 a - \sin^2 a}{R};$$

formulas which enable us to find the sine and cosine of the double arc, knowing the sine and cosine of the simple arc. This forms the problem of doubling an arc.

Reciprocally, to divide a given arc a into two equal parts, let us, in the same formulas, put $\frac{1}{2}a$ instead of a; we shall have

$$\sin a = \frac{2 \sin \frac{1}{2}a \cos \frac{1}{2}a}{R}, \quad \cos a = \frac{\cos^2 \frac{1}{2}a - \sin^2 \frac{1}{2}a}{R}.$$

Now, since we have at once $\cos^2 \frac{1}{2}a + \sin^2 \frac{1}{2} = R^2$, and $\cos^2 \frac{1}{2}a - \sin^2 \frac{1}{2}a = R \cos a$, there results

$$\cos^2 \frac{1}{2}a = \frac{1}{2}R^2 + \frac{1}{2}R \cos a, \text{ and } \sin^2 \frac{1}{2}a = \frac{1}{2}R^2 - \frac{1}{2}R \cos a;$$

whence

$$\sin \frac{1}{2}a = \sqrt{(\frac{1}{2}R^2 - \frac{1}{2}R \cos a)}$$
$$\cos \frac{1}{2}a = \sqrt{(\frac{1}{2}R^2 + \frac{1}{2}R \cos a)}.$$

Thus, making $a=100°$, or $\cos a = 0$, we have $\sin 50° = \cos 50° = \sqrt{\frac{1}{2}R^2} = R\sqrt{\frac{1}{2}}$; next, making $a=50°$, which gives $\cos a = R\sqrt{\frac{1}{2}}$, we shall have $\sin 25° = R(\sqrt{\frac{1}{2}} - \frac{1}{2}\sqrt{\frac{1}{2}})$, and $\cos 25° = R\sqrt{(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}})}$.

* XXI. The values of $\sin \frac{1}{2}a$ and $\cos \frac{1}{2}a$ may also be obtained in terms of $\sin a$; which will be useful on many occasions. These values are:

$$\sin \frac{1}{2}a = \frac{1}{2}\sqrt{(R^2 + R \sin a)} - \frac{1}{2}\sqrt{(R^2 - R \sin a)},$$
$$\cos \frac{1}{2}a = \frac{1}{2}\sqrt{(R^2 + R \sin a)} + \frac{1}{2}\sqrt{(R^2 - R \sin a)}.$$

Accordingly, by squaring the first, we shall have $\sin^2 \frac{1}{2}a = \frac{1}{4}(R^2 + R \sin a) + \frac{1}{4}(R^2 - R \sin a) - \frac{1}{2}\sqrt{(R - R^2 \sin^2 a)} = \frac{1}{2}R^2 - R \cos a$; in like manner, we should have $\cos^2 \frac{1}{2}a = \frac{1}{2}R^2 + \frac{1}{2}R \cos a$; values which agree with those already found for $\sin \frac{1}{2}a$ and $\cos \frac{1}{2}a$. It must be observed, however, that if $\cos a$ were negative, the radical $\sqrt{(R^2 - R \sin a)}$ would require to be taken with a contrary sign in the values of $\sin \frac{1}{2}a$ and $\cos \frac{1}{2}a$; and thus the one value would be changed into the other.

* XXII. By means of these formulas, it is easy to determine the sines and cosines of all the tenths of the quadrant.†

First, let $sin\, 20° = x$; $2x$ will be the chord of 40°, or the side of the regular inscribed decagon: now this side (5. IV.) is equal to the greater segment of the radius divided in extreme and mean ratio; hence making the radius $=1$, we shall have $1 : 2 :: 2x : 1 - 2x$. Hence we obtain $4x^2 = 1 - 2x$, or $x^2 + \frac{1}{2}x = \frac{1}{4}$; hence $(x+\frac{1}{4})^2 = \frac{1}{4} + \frac{1}{16} =_{5}$; hence $x + \frac{1}{4} = \frac{1}{4}\sqrt{5}$; and finally x or $sin.\, 20° = \frac{1}{4}(-1+\sqrt{5})$.

This value raised to the second power gives $sin^2\, 20° = \frac{6-2\sqrt{5}}{16}$; hence $1 - sin^2\, 20°$, or $cos^2\, 20° = \frac{10+2\sqrt{5}}{16}$. But $cos^2 a - sin^2 a = cos\, 2a$, hence $cos\, 40°$ or $sin\, 60° = \frac{4+4\sqrt{5}}{16} = \frac{1+\sqrt{5}}{4}$.

Now, if in the formulas of Art. 21. we make $R=1$, $a=20°$, and $sin\, a = \frac{1}{4}(-1+\sqrt{5})$, we shall find

$$sin\, 10° = \tfrac{1}{4}\sqrt{(3+\sqrt{5})} - \tfrac{1}{4}\sqrt{(5-\sqrt{5})}$$
$$cos\, 10° = \tfrac{1}{4}\sqrt{(3+\sqrt{5})} + \tfrac{1}{4}\sqrt{(5-\sqrt{5})}.$$

Next, if in the same formulas, we make $a=60°$ and $sin\, a = \frac{1}{4}(1+\sqrt{5})$, we shall have

$$sin\, 30° = \tfrac{1}{4}\sqrt{(5+\sqrt{5})} - \tfrac{1}{4}\sqrt{(3-\sqrt{5})}$$
$$cos\, 30° = \tfrac{1}{4}\sqrt{(5+\sqrt{5})} + \tfrac{1}{4}\sqrt{(3-\sqrt{5})}.$$

With these values, and the values of $sin\, 50°$ and $sin\, 100°$, which are already known, the following table may be formed:

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† Only according to the centesimal scale. It can be shewn, that except for the arc 3° ⅓ and its multiples, no accurate expressions of the sines can be found, even under an incommensurable form; and, as this arc 3° ⅓ on the centesimal scale corresponds to 3° on the nonagesimal, it follows that 30° and 60° are the only decimal arcs on the latter whose sines are capable of being accurately expressed, the sine of 90° being put = 1. For the multiples of this arc 3°, however, by a process exactly similar to that followed in the text, making use of the nonagesimal division, we obtain the following values:

$$sin\ \ 3° = cos\ 87° = \frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}-1) - \frac{\sqrt{3}-1}{8}\sqrt{(5+\sqrt{5})}.$$
$$sin\ \ 6° = cos\ 84° = -\frac{1}{8}(\sqrt{5}+1) + \frac{\sqrt{3}}{4\sqrt{2}}\sqrt{(5-\sqrt{5})}.$$
$$sin\ \ 9° = cos\ 81° = \frac{1}{4\sqrt{2}}(\sqrt{5}+1) - \frac{1}{4}\sqrt{(5-\sqrt{5})}.$$

$\sin\ 0^\circ = \cos 100^\circ = 0$

$\sin 10^\circ = \cos 90^\circ = \frac{1}{4}\sqrt{(3+\sqrt{5})} - \frac{1}{4}\sqrt{(5-\sqrt{5})}$

$\sin 20^\circ = \cos 80^\circ = \frac{1}{4}(-1+\sqrt{5})$

$\sin 30^\circ = \cos 70^\circ = \frac{1}{4}\sqrt{(5+\sqrt{5})} - \frac{1}{4}\sqrt{(3-\sqrt{5})}$

$\sin 40^\circ = \cos 60^\circ = \frac{1}{4}\sqrt{(10-2\sqrt{5})}$

$\sin 50^\circ = \cos 50^\circ = \frac{1}{2}\sqrt{2}$

$\sin 60^\circ = \cos 40^\circ = \frac{1}{4}(1+\sqrt{5})$

$\sin 70^\circ = \cos 30^\circ = \frac{1}{4}\sqrt{(5+\sqrt{5})} + \frac{1}{4}\sqrt{(3-\sqrt{5})}$

$\sin 80^\circ = \cos 20^\circ = \frac{1}{4}\sqrt{(10+2\sqrt{5})}$

$\sin 90^\circ = \cos 10^\circ = \frac{1}{4}\sqrt{(3+\sqrt{5})} + \frac{1}{4}\sqrt{(5-\sqrt{5})}$

$\sin 100^\circ = \cos\ 0^\circ = 1.$

$$\sin 12^\circ = \cos \overset{78^\circ}{\cancel{98}}{}^\circ = -\frac{\sqrt{3}}{8}(\sqrt{5}-1) + \frac{1}{4\sqrt{2}}\sqrt{(5+\sqrt{5})}.$$

$$\sin 15^\circ = \cos 75^\circ = \frac{1}{2\sqrt{2}}(\sqrt{3}-1).$$

$$\sin 18^\circ = \cos 72^\circ = \frac{1}{4}(\sqrt{5}-1).$$

$$\sin 21^\circ = \cos 69^\circ = -\frac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}+1) + \frac{\sqrt{3}+1}{8}\sqrt{(5-\sqrt{5})}.$$

$$\sin 24^\circ = \cos 66^\circ = \frac{\sqrt{3}}{8}(\sqrt{5}+1) - \frac{1}{4\sqrt{2}}\sqrt{(5-\sqrt{5})}.$$

$$\sin 27^\circ = \cos 63^\circ = -\frac{1}{4\sqrt{2}}(\sqrt{5}-1) + \frac{1}{4}\sqrt{(5+\sqrt{5})}.$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}.$$

$$\sin 33^\circ = \cos 57^\circ = \frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}-1) + \frac{\sqrt{3}-1}{8}\sqrt{(5+\sqrt{5})}.$$

$$\sin 36^\circ = \cos 54^\circ = \ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{2\sqrt{2}}\sqrt{(5-\sqrt{5})}.$$

$$\sin 39^\circ = \cos 51^\circ = \frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}+1) - \frac{\sqrt{3}-1}{8}\sqrt{(5-\sqrt{5})}.$$

$$\sin 42^\circ = \cos 48^\circ = -\frac{1}{8}(\sqrt{5}-1) + \frac{\sqrt{3}}{4\sqrt{2}}\sqrt{(5+\sqrt{5})}.$$

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

$$\sin 48^\circ = \cos 42^\circ = \frac{\sqrt{3}}{8}(\sqrt{5}-1) + \frac{1}{4\sqrt{2}}\sqrt{(5+\sqrt{5})}.$$

$$\sin 51^\circ = \cos 39^\circ = \frac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}+1) + \frac{\sqrt{3}+1}{8}\sqrt{(5-\sqrt{5})}.$$

These values may farther be simplified, since we have $\sqrt{(3+\sqrt{5})}=\frac{1}{2}\sqrt{10}+\frac{1}{2}\sqrt{2}$, and $\sqrt{(3-\sqrt{5})}=\frac{1}{2}\sqrt{10}-\frac{1}{2}\sqrt{2}$; from which it appears that considering $\sqrt{2}$, $\sqrt{5}$ and $\sqrt{10}$ as known quantities, there remain only four extractions of square-roots, till we arrive at the values of the sines and cosines belonging to all the arcs which are multiples of 10°.

* XXIII. From these formulas we may draw two remarkable consequences:

First, Since $2\,sin\ 40°$ is the chord of 80°, or the side of the regular inscribed pentagon, this side $=\frac{1}{2}\sqrt{(10-2\sqrt{5})}$, its square $=\frac{10-2\sqrt{5}}{4}$. The side of the regular decagon $=2\,sin\ 20°=\frac{1}{2}(-1+\sqrt{5})$, its square $=\frac{1}{4}(6-2\sqrt{5}$. Now $\frac{1}{4}(10-2\sqrt{5})=1+\frac{1}{4}(6-2\sqrt{5})$; hence *the sum formed by the square*

$sin\ 54° = cos\ 36° = \frac{1}{4}(\sqrt{5}+1).$

$sin\ 57° = cos\ 33° = -\frac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}-1)+\frac{\sqrt{3}+1}{8}\sqrt{(5+\sqrt{5})}.$

$sin\ 60° = cos\ 30° = \frac{\sqrt{3}}{2}.$

$sin\ 63° = cos\ 27° = \frac{1}{4\sqrt{2}}(\sqrt{5}-1)+\frac{1}{4}\sqrt{(5+\sqrt{5})}.$

$sin\ 66° = cos\ 24° = \frac{1}{8}(\sqrt{5}+1)+\frac{\sqrt{3}}{4\sqrt{2}}\sqrt{(5-\sqrt{5})}.$

$sin\ 69° = cos\ 21° = \frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}+1)+\frac{\sqrt{3}-1}{8}\sqrt{(5-\sqrt{5})}.$

$sin\ 72° = cos\ 18° = \ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{2\sqrt{2}}\sqrt{(5+\sqrt{5})}.$

$sin\ 75° = cos\ 15° = \frac{1}{2\sqrt{2}}(\sqrt{3}+1).$

$sin\ 78° = cos\ 12° = \frac{1}{8}(\sqrt{5}-1)+\frac{\sqrt{3}}{4\sqrt{2}}\sqrt{(5+\sqrt{5})}.$

$sin\ 81° = cos\ 9° = \frac{1}{4\sqrt{2}}(\sqrt{5}+1)+\frac{1}{4}\sqrt{(5-\sqrt{5})}.$

$sin\ 84° = cos\ 6° = \frac{\sqrt{3}}{8}(\sqrt{5}+1)+\frac{1}{4\sqrt{2}}\sqrt{(5-\sqrt{5})}.$

$sin\ 87° = cos\ 3° = \frac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}-1)+\frac{\sqrt{3}+}{8}\sqrt{(5+\sqrt{5})}.$

$sin\ 90° = cos\ 0° = 1.$

These values are from *Cagnoli's Trigonométrie*.—ED.

of the radius and the square of the decagon's side is equal to the square of the inscribed regular pentagon's side.

Secondly, Among the sines of the uneven decimal divisions of the quadrant, we have this relation:

$$\sin 90^\circ+\sin 30^\circ+\sin 10^\circ=\sin 50^\circ+\sin 70^\circ,$$

and the even divisions in like manner give $\sin 60^\circ=\sin 20^\circ+\frac{1}{2}$. These formulas, however, are only particular cases: we might shew that x being an arc measuring any number of degrees, we should have

$$\sin(100^\circ-x)+\sin(20^\circ+x)+\sin(20^\circ-x)=\sin(60-x)+\sin(60^\circ+x).$$

For, the formula $\sin(a+b)+\sin(a-b)=2\sin a\cos b$, gives

$$\sin(20^\circ+x)+\sin(20^\circ-x)=2\sin 20^\circ\cos x$$
$$\sin(60^\circ+x)+\sin(60^\circ-x)=2\sin 60^\circ\cos x.$$

Therefore, since we have $\sin 60^\circ-\sin 20^\circ=\frac{1}{2}$, and $\cos x=\sin(100^\circ-x)$, those two equations, by subtraction, will give

$$\sin(60^\circ+x)+\sin(60^\circ-x)-\sin(20+x)-\sin(20^\circ-x)=\sin(100-x).$$

A formula from which may be derived the equation of the uneven divisions by making $x=10^\circ$, and which may serve in general for verifying the tables of sines.

XXIV. If in the first and third formulas of Art. 19, we put $b=2a$, we shall have

$$\sin 3a=\frac{\sin 2a\cos a+\cos 2a\sin a}{R},\cos 3a=\frac{\cos 2a\cos a-\sin 2a\sin a}{R}.$$

Instead of $\sin 2a$ and $\cos 2a$, substitute in these expressions the values found in Art. 20, simplifying the results by means of the equation $\sin^2 a+\cos^2 a=R^2$; we shall have

$$\sin 3a=3\sin a-\frac{4\sin^3 a}{R^2},$$
$$\cos 3a=\frac{4\cos^3 a}{R^2}-3\cos a.$$

These formulas, which serve for the triplication of arcs, may likewise serve for their trisection or division into three equal parts. Thus, if we put $\sin 3a=c$, and $\sin a=x$, for determining this x we shall have the equation $cR^2=3R^2x-4x^3$. From which it appears, that the problem of trisecting an angle, when considered analytically, is of the third degree.

If in the same formulas of Art. 19, we successively put $b=3a$, $b=4a$, &c. we may obtain the sines and cosines of the arcs $4a$, $5a$, &c. that is to say, in general the sines and cosines of all the multiples of a. And conversely, the formulas, which serve for the multiplication of arcs, will give us equations to resolve for dividing a given arc into equal parts; in other words, for determining $\sin a$ or $\cos a$ when $\sin na$ and $\cos na$ are known.

* XXV. Let us farther develope the values of $\sin 5a$ and $\cos 5a$. For this purpose, we shall use the formulas

$$\sin(3a+2a)=\frac{\sin 3a\cos 2a+\cos 3a\sin 2a}{R},$$

$$\cos(3a+2a)=\frac{\cos 3a\cos 2a-\sin 3a\sin 2a}{R}.$$

Substituting here the values already found in Art. 20, and 24, after reducing, we shall have

$$\sin 5a=5\sin a-\frac{20\sin^3 a}{R^2}+\frac{16\sin^5 a}{R^4},$$

$$\cos 5a=5\cos a-\frac{20\cos^3 a}{R^2}+\frac{16\cos^5 a}{R^4}.$$

From which it appears that the problem of quintisecting an angle, or dividing it into five equal parts, would be of the fifth degree; and so likewise with regard to the other divisions by the prime numbers 7, 11, 13, &c.

* XXVI. Let it be proposed, for example, to find the approximate value of $\sin 1°$, carried to fifteen places of decimals; a value which is of use in constructing the tables of sines.† The expression $\sin 10°$, found Art. 22, being reduced to decimals on the supposition of $R=1$, gives $\sin 10°=0.15643\ 44650\ 40231$; from which, by the formula of Art. 21, we get $\sin 5°=0.07845\ 90957\ 27845$.

Now put $\sin 1°=x$; in order to obtain x, we must solve the equation

$$16x^5-20x^3+5x=0.07845\ 90957\ 27845.$$

If, for the sake of brevity, the second member is put $=c$, we shall have very nearly $5x-20x^3=c$, and $x=\frac{1}{5}c+4\left(\frac{1}{5}c\right)^3$. Now $\frac{1}{5}c=0.01569\ 18191$ and $4\left(\frac{1}{5}c\right)^3=0.00001\ 5456$; hence for the first approximation, we have $x=0.01570\ 7275$, a value which is accurate to the eighth place of decimals. In order to find a more accurate value, put $x=0.01570\ 7275$; by substituting this in the proposed equation, neglecting the squares and the other powers of y, we shall have

$$0.0784590009424927+4.9852017\,y=0.0784590095727845;$$

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† According to the nonagesimal scale, the value of $\sin 1°$ is found still more readily, by applying the formula in Art. 24. to the value of $\sin 3°$ given in the last note. It is equal to 0.0174524064.—Ed.

whence we obtain y=0.00000 00 173 118207, and

x or sin 1°=0.01570 73173 118207.

From the sines of 1° or 100′, we could by the same process derive the sines of 50′, of 10′, of 5′, and finally of 1′.

XXVII. The formulas of Art. 19, furnish a great number of consequences; among which it will be enough to mention those of most frequent use. The four which follow,

$$sin\ a\ cos\ b=\tfrac{1}{2}\ \mathrm{R}\ sin\ (a+b)+\tfrac{1}{2}\ \mathrm{R}\ cin(a-b)$$

$$sin\ b\ cos\ a=\tfrac{1}{2}\ \mathrm{R}\ sin\ (a+b)-\tfrac{1}{2}\ \mathrm{R}\ sin(a-b)$$

$$cos\ a\ cos\ b=\tfrac{1}{2}\ \mathrm{R}\ cos\ (a-b)+\tfrac{1}{2}\ \mathrm{R}\ cos(a+b)$$

$$sin\ a\ sin\ b=\tfrac{1}{2}\ \mathrm{R}\ cos\ (a-b)-\tfrac{1}{2}\ \mathrm{R}\ cos(a+b)$$

serve to change a product of several sines or cosines into *linear* sines or cosines, that is, into sines and cosines multiplied only by constant quantities.

XXVIII. If in these formulas we put $a+b=p$, $a-b=q$, which gives $a\frac{p+q}{2}$, $b=\frac{p-q}{2}$, we shall find

$$sin\ p+sin\ q=\frac{2}{\mathrm{R}}\ sin\ \tfrac{1}{2}\ (p+q)\ cos\ \tfrac{1}{2}\ (p-q)$$

$$sin\ p-sin\ q=\frac{2}{\mathrm{R}}\ sin\ \tfrac{1}{2}\ (p-q)\ cos\ \tfrac{1}{2}\ (p+q)$$

$$cos\ p+cos\ q=\frac{2}{\mathrm{R}}\ cos\ \tfrac{1}{2}\ (p+q)\ cos\ \tfrac{1}{2}\ (p-q)$$

$$cos\ q-cos\ p=\frac{2}{\mathrm{R}}\ sin\ \tfrac{1}{2}\ (p+q)\ sin\ \tfrac{1}{2}\ (p-q)\ ;$$

new formulas, which are often employed in trigonometrical calculations for reducing two terms to a single one.

XXIX. Finally, from these latter formulas, by dividing, and considering that $\frac{sin\ a}{cos\ a}=\frac{tang\ a}{\mathrm{R}}=\frac{\mathrm{R}}{cot\ a}$, we derive the following:

$$\frac{sin\ p+sin\ q}{sin\ p-sin\ q}=\frac{sin\tfrac{1}{2}(p+q)cos\tfrac{1}{2}(p-q)}{cos\tfrac{1}{2}(p+q)sin\tfrac{1}{2}(p-q)}=\frac{tang\tfrac{1}{2}(p+q)}{tang\tfrac{1}{2}(p-q)}$$

$$\frac{sin\ p+sin\ q}{cos\ p+cos\ q}=\frac{sin\tfrac{1}{2}(p+q)}{cos\tfrac{1}{2}(p+q)}=\frac{tang\tfrac{1}{2}(p+q)}{\mathrm{R}}$$

$$\frac{sin\ p+sin\ q}{cos\ q-cos\ p}=\frac{cos\tfrac{1}{2}(p-q)}{sin\tfrac{1}{2}(p-q)}=\frac{cot\tfrac{1}{2}(p-q)}{\mathrm{R}}$$

$$\frac{\sin p-\sin q}{\cos p+\cos q}=\frac{\sin\frac{1}{2}(p-q)}{d)\frac{1}{4}soc-q)}=\frac{\text{tang}\frac{1}{2}(p-q)}{R}$$

$$\frac{\sin p-\sin q}{\cos q-\cos p}=\frac{\cot\frac{1}{2}(p+q)}{\sin\frac{1}{2}(p+q)}=\frac{\cot\frac{1}{2}(p+q)}{R}$$

$$\frac{\cos p+\cos q}{\cos q-\cos p}=\frac{\cos\frac{1}{2}(p+q)}{\sin\frac{1}{2}(p+q)}\ \frac{\cos\frac{1}{2}(p-q)}{\sin\frac{1}{2}(p-q)}=\frac{\cot\frac{1}{2}(p+q)}{\text{tang}\frac{1}{2}(p-q)}$$

$$\frac{\sin\ (p+q)}{\sin p+\sin q}=\frac{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)}{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}=\frac{\cos\frac{1}{2}(p+q)}{\cos\frac{1}{2}(p-q)}$$

$$\frac{\sin\ (p+q)}{\sin p-\sin q}=\frac{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)}{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)}=\frac{\sin\frac{1}{2}(p+q)}{\sin\frac{1}{2}(p-q)}$$

Formulas which are the expression of so many theorems. From the first, it follows that *the sum of the sines of two arcs is to the difference of these sines, as the tangent of half the sum of the arcs is to the tangent of half their difference.*

XXX. Making $b=a$, or $q=0$, in the formulas of the three preceding Articles, we shall have the following results:

$$\cos^2 a=\tfrac{1}{2}R^2+\tfrac{1}{2}R\cos 2a$$

$$\sin^2 a=\tfrac{1}{2}R^2-\tfrac{1}{2}R\cos 2a$$

$$R+\cos p=\frac{2\cos^2\frac{1}{2}p}{R}$$

$$R-\cos p=\frac{2\sin^2\frac{1}{2}p}{R}$$

$$\sin p=\frac{2\sin\frac{1}{2}p\cos\frac{1}{2}p}{R}$$

$$\frac{\sin p}{R+\cos p}=\frac{\text{tang}\frac{1}{2}p}{R}=\frac{R}{\cot\frac{1}{2}p}$$

$$\frac{\sin p}{R-\cos p}=\frac{\cot\frac{1}{2}p}{R}=\frac{R}{\text{tang}\frac{1}{2}p}$$

$$\frac{R+\cos p}{R-\cos p}=\frac{\cot^2\frac{1}{2}p}{R^2}=\frac{R^2}{\text{tang}^2\frac{1}{2}p}.$$

XXXI. In order likewise to develope some formulas relative to tangents, let us consider the expression

$\text{tang}\ (a+b)=\dfrac{R\sin\ (a+b)}{\cos\ (a+b)}$, in which, by substituting the values of $\sin\ (a+b)$ and $\cos\ (a+b)$, we shall find

$$\text{tang}\ (a+b)=\frac{R\ (\sin a\cos b+\sin b\cos a)}{\cos a\cos b-\sin b\sin a}.$$

Now we have $\sin a = \frac{\cos a \tang a}{R}$, and $\sin b = \frac{\cos b \tang b}{R}$;
substituting these values, dividing all the terms by $\cos a \cos b$; we shall have

$$\tang (a+b) = \frac{R^2 (\tang a + \tang b)}{R^2 - \tang a \tang b},$$

which is the value of the tangent of the sum of two arcs, expressed by the tangents of each of these arcs. For the tangent of their difference, we should in like manner find

$$\tang (a-b) = \frac{R^2 (\tang a - \tang b)}{R^2 + \tang a \tang b}.$$

Suppose $b=a$; for the duplication of the arcs, we shall have the formula

$$\tang 2a = \frac{2 R^2 \tang a}{R^2 - \tang^2 a},$$

whence would result

$$\cot 2a = \frac{R^2}{\tang 2a} = \frac{R^2 - \tang^2 a}{2 \tang a} = \tfrac{1}{2} \cot a - \tfrac{1}{2} \tang a.$$

Suppose $b = 2a$; for their triplication, we shall have the formula

$$\tang 3a = \frac{R^2 (\tang a + \tang 2a)}{R^2 - \tang a \tang 2a};$$

in which, substituting the value of $\tang 2a$, we shall have

$$\tang 3a = \frac{3 R^2 \tang a - \tang^3 a}{R^2 - 3 \tang^2 a}.$$

* XXXII. The developement of trigonometrical formulas, considered in all its extent, forms an important branch of Analysis, on which may be consulted the excellent work of Euler, entitled *Introductio in Analysin Infinitorum*, or the French translation of it, illustrated with notes, by M. Labey. It seems proper, notwithstanding, that we should here demonstrate the formulas which serve for expressing the sines and cosines as functions of the arc; both because these formulas are referred to as known in Note IV. and because they are required in constructing the tables.

First, then, supposing the radius $=1$, which does not affect the universality of the results, we have the formula $\cos^2 A + \sin^2 A = 1$, the first member of which may be regarded as the product of the two imaginary factors $\cos A + \sqrt{-1} \sin A$, and $\cos A - \sqrt{-1} \sin A$. Multiplying together the two similar factors $\cos A+\sqrt{-1}\sin A$, $\cos B+\sqrt{-1}\sin B$, the product will be $\cos A\cos B-\sin A\sin B+(\sin A\cos B+\sin B\cos A)\sqrt{-1}$, and may consequently be reduced to the form $\cos(A+B)+\sqrt{-1}\sin(A+B)$, which is similar to each of the factors. Hence generally we have

$$(\cos A+\sqrt{-1}\sin A)(\cos B+\sqrt{-1}\sin B)=\cos(A+B)+\sqrt{-1}\sin(A+B):$$

and it is worthy of remark, that multiplication among quantities of this kind is performed by simply adding the arcs; a property analogous to that of logarithms. We shall successively obtain

$$(\cos A+\sqrt{-1}\sin A)(\cos A+\sqrt{-1}\sin A)=\cos 2A+\sqrt{-1}\sin 2A$$
$$(\cos A+\sqrt{-1}\sin A)(\cos 2A+\sqrt{-1}\sin 2A)=\cos 3A+\sqrt{-1}\sin 3A$$
$$(\cos A+\sqrt{-1}\sin A)(\cos 3A+\sqrt{-1}\sin 3A)=\cos 4A+\sqrt{-1}\sin 4A$$
&c.

The first product is equal to $(\cos A+\sqrt{-1}\sin A)^2$, the second is equal to $(\cos A+\sqrt{-1}\sin A)^3$; and so on. Hence in general, n being any whole number, we shall have

$$(\cos A+\sqrt{-1}\sin A)^n=\cos nA+\sqrt{-1}\sin nA.$$

From this, by changing the sign of $\sqrt{-1}$, there results

$$(\cos A-\sqrt{-1}\sin -A)^n=\cos nA-\sqrt{-1}\sin nA,$$

and out of these two equations, which follow from each other, the values of $\sin nA$ and $\cos nA$ may be separately deduced, namely:

$$\cos nA=\tfrac{1}{2}(\cos A+\sqrt{-1}\sin A)^n+\tfrac{1}{2}(\cos A-\sqrt{-1}\sin A)^n$$

$$\sin nA=\frac{1}{2\sqrt{-1}}(\cos A+\sqrt{-1}\sin A)^n-\frac{1}{2\sqrt{-1}}(\cos A-\sqrt{-1}\sin A)^n$$

* XXXIII. If the same quantities are to be expressed in series, we have only by means of the binomial theorem, to develope $(\cos A+\sqrt{-1}\sin A)^n$, which will give

$$\cos^n A+\frac{n}{1}\cos^{n-1}A\sin A\sqrt{-1}-\frac{n.n-1}{1.2}\cos^{n-2}A\sin^2 A-$$

$$\frac{n.n-1.n-2}{1.2.3.}\cos^{n-3}A\sqrt{-1}+\frac{n.n-1.n-2.n-3}{1.2.3.4.}\cos^{n-4}A\sin^4 A+\&c$$

And this quantity being the value of $\cos nA+\sqrt{-1}\sin nA$, we may separately put the real part of it equal to $\cos nA$, and the imaginary part equal to $\sqrt{-1}\sin nA$. We shall then have

$$\cos nA=\cos^n A-\frac{n.n-1}{1.2}\cos^{n-2}A\sin^2 A+\frac{n.n-1.n-2.n-3}{1.2.3.4.}$$
$$\cos^{n-4}A\sin^4 A-\&c.$$

$$\sin nA=n\cos^{n-1}A\sin A-\frac{n.n-1.n-2}{1.2.3.}\cos^{n-3}A\sin^3 A+\&c.$$

two series whose law it is easy to discover, and by means of which the sine and cosine of a multiple arc A may be found, in a much speedier manner than by the operations indicated in Art. 24.

* XXXIV. Since we have *sin* A = *cos* A *tang* A, these two series may be put under the form

$$cos\ nA = cos^n A\left(1 - \frac{n.n-1.}{1.2} tang^2 A + \frac{n.n-1.n-2.n-3}{1.2.3.4.} tang^4 A - \&c.\right)$$

$$sin\ nA = cos^n A\left(\frac{n}{1} tang\ A - \frac{n.n-1.n-2}{1.2.3} tang^3 A + \&c.\right)$$

Suppose $n = \frac{x}{A}$; by substituting this value, and still retaining factor $cos^n A$, we shall have

$$cos\ x = cos^n A\left(1 - \frac{x.x-A}{1.2} . \frac{tang^2 A}{A^2} + \frac{x.x-A.x-2A.x-3A}{1.2.3.4} . \frac{tang^4 A}{A^4} - \&c.\right)$$

$$sin\ x = cos^n A\left(\frac{x}{1} . \frac{tang\ A}{A} - \frac{x.x-A.x-2A}{1.2.3} . \frac{tang^3 A}{A^3} + \&c.\right)$$

In these formulas A may be assumed at pleasure: suppose A very small; then $\frac{tang\ A}{A}$ will be very little different from unity, the tangent of a very small arc being almost equal to the arc. Yet so long as the arc is not absolutely nothing, we have $tang\ A > A$* or $\frac{tang\ A}{A} > 1$; we have at the same time $A > sin\ A$†; hence $\frac{tang\ A}{A} < \frac{tang\ A}{sin\ A}$, or $\frac{tang\ A}{A} < \frac{1}{cos\ A}$. Hence it appears that the ratio $\frac{tang\ A}{A}$ is always included between the limits 1 and $\frac{1}{cos\ A}$. Put $A = 0$; we shall have $cos\ A = 1$: hence because $\frac{tang\ A}{A}$ is included between 1 and $\frac{1}{cos\ A}$, we shall have exactly $\frac{tang\,A}{A} = 1$. Therefore making $A=0$, we shall have

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* AT (see the fig. of page 285) is greater than AM, because the triangle ATC is to the sector ACM : S AT × ½ AC : AM × ½ AC : : AT : AM.

† AM (see the same fig.) is greater than MP, because the arc MAN is greater than its chord MN.

$$cos\,x = cos^n A\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.\right)$$

$$sin\,x = cos^n A\left(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.\right)$$

It remains to inquire what becomes of cos^n A, supposing A to diminish more and more, and at length to vanish. Now we have $\frac{1}{cos^2 A} = sec^2 A = +tang^2 A$; hence $cos\,A = (1 + tang^2 A)^{-\frac{1}{2}}$,

hence $cos^n A = (1 + tang^2 A)^{-\frac{n}{2}} = 1 - \frac{2}{n} tang^2 A + \frac{n.n+2}{2.4} tang^4 A$ &c.

Instead of n, substitute its value $\frac{x}{A}$; we shall have

$$cos^n A = 1 - \frac{x}{2} A . \frac{tang^2 A}{A^2} + \frac{x . x+2A}{2.4} A^2 . \frac{tang^4 A}{A^4} - \&c.$$

If we now suppose A to diminish more and more, while x remains the same, the value of cos^n A will more and more approach to unity; and in fine, if A is made $=0$ and $\frac{tang\,A}{A} = 1$, we shall have exactly $cos^n A = 1$. Hence we have the formulas

$$cos\,x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.$$

$$sin\,x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$$

by means of which, we are enabled to calculate the sine and the cosine of an arc, whose length is given, in parts of the radius taken as unit.

* XXXV. These same values may be expressed more briefly, by means of exponentials. For this purpose, we must consider that e being the number whose hyperbolic logarithm is 1, we have

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$$

If, in this formula, we make $x = z\sqrt{-1}$, we shall have

$$e^{x\sqrt{-1}} = 1 + \frac{x\sqrt{-1}}{1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \&c.$$

By changing the sign of $\sqrt{-1}$, we should in like manner have

$$e^{-x\sqrt{-1}}=1-\frac{x\sqrt{-1}}{1}-\frac{x^2}{1.2}+\frac{x^3\sqrt{-1}}{1.2.3}+\frac{x^4}{1.2.3.4}-\frac{x\sqrt{-1}}{1.2.3.4.5}-\&c.$$

Hence we deduce

$$\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\&c.$$

$$\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}=x-\frac{x}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&c.$$

two series, whose second members are the values already found for *cos x* and *sin x*. Hence we have

$$cos\ x=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2},\ sin\ x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}};$$

whence we find $\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}=\sqrt{-1}\,.\,\frac{sin\ x}{cos\ x}=\sqrt{-1}\,tang\,x,$

the formula employed in Note IV.

The same formulas give $e^{x\sqrt{-1}}=cos\ x+\sqrt{-1}\ sin\ x$, $e^{-x\sqrt{-1}}=cos\ x-\sqrt{-1}\ sin\ x$; hence dividing the one by the other, we shall have $e^{2x\sqrt{-1}}=\frac{cos\ x+\sqrt{-1}\ sin\ x}{cos\ x-\sqrt{-1}\ sin\ x}=\frac{1+\sqrt{-1}\ tang\ x}{1-\sqrt{-1}\ tang\ x}$, or taking the logarithms of each member, $2x\sqrt{-1}=\log.\left(\frac{1+\sqrt{-1}\ tang\ x}{1-\sqrt{-1}\ tang\ x}\right)$. But we already know that $\log.\left(\frac{1+z}{1-z}\right)=2z+\frac{2}{3}z^3+\frac{2}{5}z^5+\&c.$; hence putting $\sqrt{-}$ *tang x* instead of z, and dividing both sides by $2\sqrt{-1}$, we shall have

$$x=tang\ x-\tfrac{1}{3}\,tang^3\,x+\tfrac{1}{5}\,tang^5\,x-\tfrac{1}{7}\,tang^7\,x+\&c.$$

A very simple formula, which enables us to calculate the arc from its tangent when the latter is less than unity.

* XXXVI. In order to apply the preceding formulas for determining the sine and cosine of an arc given in degrees and parts of a degree, we must have the length of this arc expressed in parts of the radius, in other words, we must have the ratio of this arc to the radius. Now, the radius being 1, the semicircumference or the arc of 200° = 3. 14159 26535 897932.

Put this number $= \pi$, the length of the arc $\frac{m}{n}\,100°$ will be $\frac{m}{n} \cdot \frac{\pi}{2}$; hence, if in the preceding formulas, we make $x = \frac{m}{n} \cdot \frac{\pi}{2}$, and afterwards insert the value of π, calculating the coefficients to sixteen places of decimals, we shall have the following formulas:

$sin\left(\frac{m}{n} \cdot 100°\right) =$ †	$cos\left(\frac{m}{n} \cdot 100°\right) =$
$1.57079\ 63267\ 948966\frac{m}{n}$	$1.00000\ 00000\ 000000$
$-0.64596\ 40975\ 062463\frac{m^3}{n^3}$	$-1.23370\ 05501\ 361698\frac{m^2}{n^2}$
$+0.07969\ 26262\ 461670\frac{m^5}{n^5}$	$+0.25366\ 95079\ 010480\frac{m^4}{n^4}$
$-0.00468\ 17541\ 353187\frac{m^7}{n^7}$	$-0.02086\ 34807\ 633530\frac{m^6}{n^6}$
$+0.00016\ 04411\ 847874\frac{m^9}{n^9}$	$+0.00091\ 92602\ 748394\frac{m^8}{n^8}$
$-0.00000\ 35988\ 432352\frac{m^{11}}{n^{11}}$	$-0.00002\ 52020\ 423731\frac{m^{10}}{n^{10}}$
$+0.00000\ 00569\ 217292\frac{m^{13}}{n^{13}}$	$+0.00000\ 04710\ 874779\frac{m^{12}}{n^{12}}$
$-0.00000\ 00006\ 688035\frac{m^{15}}{n^{15}}$	$-0.00000\ 00063\ 866031\frac{m^{14}}{n^{14}}$
$+0.00000\ 00000\ 060669\frac{m^{17}}{n^{17}}$	$+0.00000\ 00000\ 656596\frac{m^{16}}{n^{16}}$
$-0.00000\ 00000\ 000438\frac{m^{19}}{n^{19}}$	$-0.00000\ 00000\ 005294\frac{m^{18}}{n^{18}}$
$+0.00000\ 00000\ 000000\frac{m^{21}}{n^{21}}$	$+0.00000\ 00000\ 000034\frac{m^{20}}{n^{20}}$

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† By merely changing 100° into 90°, this table might also be applied to the nonagesimal scale; though it would be less convenient for obtaining the sines of integer arcs, because n could not then so often be put equal to some power of 10.—ED.

The sines and cosines of the arcs between 0° and 50° comprehend the cosines and sines of the arcs between 50° and 100°; for we have

$$\sin(50^\circ + z) = \cos(50^\circ - z) \text{ and } \cos(50^\circ + z) = \sin(50^\circ - z).$$

Hence in the formulas exhibiting the values of $\sin \frac{m}{n} 100^\circ$, and $\cos \frac{m}{n} 100^\circ$, we may always suppose $\frac{m}{n} \angle \frac{1}{2}$; by which means the series will converge so rapidly, that we shall never have need to calculate more than a small number of terms, especially when not many decimals are required.

By successively making $\frac{m}{n} = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}$, we shall find the following results:

$\sin 10^\circ = \cos 90^\circ = 0.15643\ 44650\ 40231$
$\sin 20^\circ = \cos 80^\circ = 0.30901\ 69943\ 74947$
$\sin 30^\circ = \cos 70^\circ = 0.45399\ 04997\ 39547$
$\sin 40^\circ = \cos 60^\circ = 0.58778\ 52522\ 92473$
$\sin 50^\circ = \cos 50^\circ = 0.70710\ 67811\ 86548$
$\sin 60^\circ = \cos 40^\circ = 0.80901\ 69943\ 74947$
$\sin 70^\circ = \cos 30^\circ = 0.89100\ 65241\ 88368$
$\sin 80^\circ = \cos 20^\circ = 0.95105\ 65162\ 95154$
$\sin 90^\circ = \cos 10^\circ = 0.98768\ 83405\ 95138$
$\sin 100^\circ = \cos 0^\circ = 1.00000\ 00000\ 00000$

which agree with the algebraical formulas of Art. 22. Also by making $\frac{m}{n} = \frac{1}{100}$, we shall find the same value for $\sin 1^\circ$ as was found in Art. 26; and the extreme facility with which these results are obtained is a proof of the excellence of the method.

ON THE CONSTRUCTION OF TABLES OF SINES.

* XXXVII. Those useful mathematicians, to whom we are indebted for the first construction of tables of sines, founded their calculations on methods which are ingenious in themselves, but difficult in their application. In subsequent times, Analysis has furnished us with far more expeditious methods of attaining this object; but the calculations being already performed, these methods would have remained without applica-

tion, if the establishment of the French metrical system had not required the calculation of new tables corresponding to the decimal division of the circle.

To give some idea of the methods proper to be followed in the construction of tables, suppose we are required to calculate the sines of all the arcs from minute to minute, which lie between 1 minute and 10000 minutes or 100 degrees. We shall make the radius $=1$, the arc of one minute $=a$; and in the first place we shall have to find the sine and cosine of the arc a with a great degree of accuracy.

The radius being 1, the semicircumference or the arc of 200 degrees, we already know is 3.14159 26535 897932; dividing this number by 20000, we have the arc of 1 minute or $a=0.00015\ 70796\ 32679\ 48966$, a value which is correct to the twentieth decimal. When an arc is very small, its sine may be considered as almost equal to the arc: hence we have very nearly $\sin a = 0.00015\ 70796\ 32679\ 48966$. But this value is wrong from the thirteenth decimal, which is only the tenth significant figure. To obtain a more correct value, the simplest method is to have recourse to the formulas in Art. 36, by which, making $\frac{m}{n}=\frac{1}{10000}$, from the first two or three terms of each series, we shall immediately have

$$\sin a = 0.00015\ 70796\ 32033\ 525563,$$
$$\cos a = 0.99999\ 99876\ 62994\ 52400\ 5253,$$

values, which are correct to the twentieth decimal for the *sine*, and to the twenty-fourth for the *cosine*.

*XXXVIII. Knowing the *sine* and the *cosine* belonging to the arc of one minute designated by a, in order to obtain from it successively the sines of all the arcs which are multiples of a, we shall, in the formulas of Art. 22, make $p=x+a$, $q=x-1$. By this substitution, R being always 1, the first and the third formulas will give

$$\sin(x+a) = 2\cos a \sin x - \sin(x-a)$$
$$\cos(x+a) = 2\cos a \cos x - \cos(x-a)$$

From these formulas it appears, that if a series of arcs are in arithmetical progression, the difference being a, their sines will form a recurring series, whose scale of relation will be $2\cos a$, -1; in other words, two consecutive sines A and B being calculated, the following sine C will be found by multiplying B by $2\cos a$, A by -1, and adding the two products, which will give $C = 2B\cos a - A$. The cosines of these arcs will in like manner form a recurring series, whose scale of relation will be $2 \cos a, -1$; hence we shall have in succession,

$\sin 0 = 0$	$\cos 0 = 1$
$\sin a = \sin a$	$\cos a = \cos a$
$\sin 2a = 2 \cos a \sin a$	$\cos 2a = 2 \cos a \cos a - 1$
$\sin 3a = 2 \cos a \sin 2a - \sin a$	$\cos 3a = 2 \cos a \cos 2a - \cos a$
$\sin 4a = 2 \cos a \sin 3a - \sin 2a$	$\cos 4a = 2 \cos a \cos 3a - \cos 2a$
$\sin 5a = 2 \cos a \sin 4a - \sin 3a$	$\cos 5a = 2 \cos a \cos 4a - \cos 3a$
&c.	&c.

* XXXIX. All that now remains is to execute the operations here indicated, by substituting the values of $\sin a$ and $\cos a$. If our tables of sines are to be constructed with ten decimal places, it will be enongh to take the values of $\sin a$ and $\cos a$, carried to 16 places, namely,

$$\sin a = 0.00015\ 70796\ 320335,$$
$$\cos a = 0.99999\ 99876\ 629945.$$

But as $\cos a$ differs very little from unity, there is a method of abbreviation, of which it will be right to take advantage. Put $k = 2\,(-\cos a) = 0.00000\ 00246\ 740110$; we shall have $2 \cos a = 2 - k$, which will give

$$\sin (x+a) - \sin x = \sin x - \sin (x-a) - k \sin x,$$
$$\cos (x+a) - \cos x = coc\ x - \cos (x-a) - k \cos x.$$

To obtain the term $\sin (x+a)$, we have only to increase the preceding term $\sin x$ by the difference $\sin (x+a) - \sin x$, which will in every case be very small. Now by the formula, this difference is equal to a similar difference already calculated, namely, $\sin x - \sin (x-a)$ *minus* the product of $\sin x$ by the constant number k. The finding of this product is thus the only operation of any length connected with the deducing of a sine from the two preceding ones. We may also observe, *first*, that this product need not be carried beyond the sixteenth place of decimals, which gives only a small number of ciphers to calculate; *secondly*, that these multiplications may be greatly abridged by finding beforehand the products of the constant number 246740110 by 1, 2, 3, &c. to 9; by which means, we shall immediately obtain the partial products resulting from the different ciphers of the multiplier $\sin x$, and nothing will remain for us but to add these products together, rejecting all the decimals beyond the sixteenth.

The same process will require to be followed in calculating the cosines; and after carrying the series of both these quantities to 50°, the table will be complete.

* XL. It is requisite, we say once more, to calculate sines with 16 decimals, that is, with five or six decimals beyond what we

mean to retain in the tables, in order to make sure that no errors, though augmented in the course of 5000 operations, may influence the tenth decimal in the concluding results. When the calculation is finished, the superfluous decimals may be struck off, and only ten retained in the table.

It is evident enough, that in executing so many calculations, we should endeavour to verify our results as often as possible. In the example, which we have exhibited, of a table calculated from minute to minute, it would be requisite to calculate beforehand the sines and cosines from degree to degree, which would afford a very useful verification at the end of every 100 terms. Now, for calculating the sines from degree to degree, we have the formulas and values which follow :

$$\sin(x+1^\circ)-\sin x=\sin x-\sin(x-1^\circ)-h\sin x,$$
$$\cos(x+1^\circ)-\cos x=\cos x-\cos(x-1^\circ)-h\cos x,$$
$$\sin 1^\circ=0.01570\ 73173\ 11820\ 676,$$
$$\cos 1^\circ=0.99987\ 66324\ 81660\ 599,$$
$$h=2(1-\cos 1^\circ)=0.00024\ 67350\ 36678\ 802.$$

The sines, thus calculated from degree to degree, will themselves be verified at every tenth term, by the already known values of *sin* 10°, *sin* 20°, &c. And finally, when the whole table is constructed, it may farther be verified in as many places as we choose, by the equation

$$\sin(100^\circ-x)+\sin(20^\circ-x)+\sin(20+x)=\sin(60^\circ-x)+\sin(60^\circ-x).$$

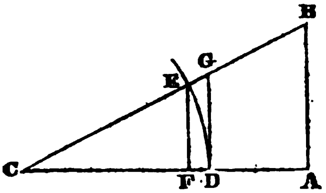
*XLI. The sines, as they result from the calculations here explained, are expressed in parts of the radius, and called *natural sines* : but in practice it has been found, that great advantages result from using the logarithms of those sines instead of the sines themselves ; and accordingly our tables for the most part do not exhibit the natural sines at all, but merely their logarithms. It is easy to conceive how, when the sines are calculated, their logarithms may be found ; but as the supposition of radius =1 would make the logarithms of all the sines negative, it has been thought proper to take radius =10000000000, in other words, to multiply by 10000000000 all the sines found on the supposition of radius =1. By this arrangement, the radius or the sine of 100°, which is often met with in calculation, has 10 units for its logarithm ; and scarcely any angle can occur in practice so small as to have a negative logarithm for its sine.

The logarithmic sines being found, the logarithms of the tangents are easily deduced from them, by a series of operations in simple subtraction : for, having $\tan g\, x=\frac{R\sin x}{\cos x}$, we must also have $log.\ tang\ x = 10 + log.\ sin\ x - log.\ cos\ x$. As to the logarithms of the secants, they may be found in a manner still more simple, by help of the equation $sec\ x = \frac{R}{cos\ x}$. It is because these deficiencies may be so easily supplied, that in the tables, no logarithms are inserted but those of the sines and tangents.

We might now explain the species of interpolation usually employed for finding the logarithmic sines or tangents of such arcs as contain fractions of a minute; or for finding the arc which corresponds to a given logarithmic sine or tangent, when the latter lies between two numbers given in the table. But for these details it is better to consult the explanations, with which the tables are always accompanied.

PRINCIPLES FOR THE SOLUTION OF RECTILINEAL TRIANGLES.

XLII. *In all right-angled triangles, the radius is to the sine of one of the acute angles, as the hypotenuse is to the side opposite this angle.*



Let ABC be the proposed triangle right-angled at A: from the point C as a centre, with a radius CD equal to the radius of the tables, describe the arc DE, which will measure the angle C; on CD let fall the perpendicular EF, which will be the sine of the angle C. The triangles CBA, CEF are similar, and give the proportion CE : EF : : CB : BA; hence

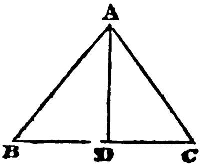
$$R : sin\ C : : BC : BA.$$

XLIII. *In all right-angled triangles, radius is to the tangent of one of the acute angles as the side lying adjacent to this angle is to the side lying opposite.*

Having described the arc DE (see the last figure), as in the preceding Article, draw DG perpendicular to CD; it will be the tangent of the angle C. From the similar triangles CDG, CAB, we shall have the proportion CD : DG : CA : AB; hence

$$R : tang\ C : : CA : AB.$$

XLIV. *In any rectilineal triangle, the sines of the angles are as the opposite sides.*



Let ABC be the proposed triangle; AD the perpendicular, let fall from the vertex A on the opposite side BC: there may be two cases.

First. If the perpendicular falls within the triangle ABC, the right-angled triangles ABD, ACD (Art. 42.) will give

$$R : \sin B :: AB : AD.$$
$$R : \sin C :: AC : AD.$$

In these two proportions, the extremes are equal; hence with the means we shall have

$$\sin C : \sin B :: AB : AC.$$

Secondly. If the perpendicular falls without the triangle ABC, (see the fig. in the next page) the right-angled triangles ABD, ACD will still give the proportions

$$R : \sin ABD :: AB : AD,$$
$$R : \sin C \quad :: AC : AD;$$

from which we derive $\sin C : \sin ABD :: AB : AC$. But the angle ABD is the supplement of ABC or B; hence $\sin ABD = \sin B$; hence we again have

$$\sin C : \sin B :: AB : AC.$$

XLV. *In all rectilineal triangles, the cosine of one angle is to radius, as the sum of the squares of the sides which contain that angle,* minus *the square of the third side is to twice the rectangle of the former two sides; in other words, we have*

$$\cos B : R :: AB^2 + BC^2 - AC^2 : 2AB \,.\, BC, \text{ or}$$

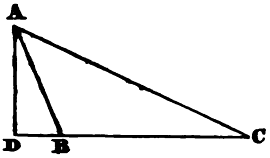
$$\cos B = R \times \frac{AB^2 + BC^2 - AC^2}{2AB \,.\, BC}.$$

From the vertex A, let AD be again drawn perpendicular to the side BC.

First. If this perpendicular falls within the triangle (see the preceding figure) we shall have $AC^2 = AB^2 + BC^2 - 2BC \times BD$ (Prop. 31. II.); hence $BD = \frac{AB^2 + BC^2 - AC^2}{2BC}$. But in the right-angled triangle ABD, we have $R : \sin BAD :: AB : BD$; also the angle BAD being the complement of B, we have *sin* BAD=*cos* B; hence $\cos B=\frac{R\times BD}{AB}$, or by substituting the value of BD,

$$\cos B=R\times\frac{AB^2+BC^2-AC^2}{2AB\times BC}.$$

Secondly. If the perpendicular falls without the triangle, we shall have $AC^2=AB^2+BC^2+2BC\times BD$ (Prop. 13. III.); hence $BD=\frac{AC^2-AB^2-BC^2}{2BC}$. But in the right-angled triangle BAD,



we still have *sin* BAD or $\cos ABD=\frac{R\times BD}{AB}$; and the angle ABD being supplemental to ABC or B, we have $\cos B=-\cos ABD=-\frac{R\times BD}{AB}$ (Art. 11.); hence by substituting the value of BD, we shall again have

$$\cos B=R\times\frac{AB^2+BC^2-AC^2}{2AB\times BC}.$$

* XLVI. Let A, B, C be the three angles of any triangle; *a*, *b*, *c* the sides respectively opposite them: by the last Article, we shall have $\cos B=R\times\frac{a^2+c^2-b^2}{2ac}$. And the same principle when applied to each of the other two angles, will, in like manner give $\cos A=R\times\frac{b^2+c^2-a^2}{2bc}$, and $\cos C=R\times\frac{a^2+b^2-c^2}{2ab}$.

These three formulas are of themselves sufficient for solving all the problems of rectilineal trigonometry; because, when three of the six quantities A, B, C, *a*, *b*, *c*, are given, we have by these formulas the equations necessary for determining the other three. The principles already explained, and whatever others may be added to them, can, therefore, only be consequences of these three principal formulas.

Accordingly the value of *cos* B gives

$$\sin^2 B=R^2-\cos^2 B=R^2.\frac{4a^2c^2-(a^2+c^2-b^2)^2}{4a^2c^2}=$$

$$\frac{R^2}{4a^2c^2}(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4):$$ hence

$$\frac{\sin B}{b}=\frac{R}{2abc}\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}.$$

The second member being a function of *a*, *b*, *c*, in which these three letters all occur under the very same form, we may evidently change two of these letters at will, and thus have $\frac{\sin B}{b}=\frac{\sin A}{a}=\frac{\sin C}{c}$; which is the principle of Art. 44. And from this, the principles of Art. 42 and 43 are easily deducible.

XLVII. *In any rectilineal triangle, the sum of two sides is to their difference, as the tangent of half the sum of the angles opposite those sides is to the tangent of half the difference of those same angles.*

From the proportion $AB : AC :: \sin C : \sin B$ (see the figures in pp. 307, 308.); we derive $AC+AB : AC-AB :: \sin B+\sin C : \sin B-\sin C$. But, according to the formulas of Art. 29, we hav

$$\sin B+\sin C : \sin B-\sin C :: \operatorname{tang}\frac{B+C}{2} : \operatorname{tang}\frac{B-C}{2};$$

hence

$$AC+AB : AC-AB :: \operatorname{tang}\frac{B+C}{2} : \operatorname{tang}\frac{B-C}{2};$$

which is the property we had to demonstrate.

With this small number of principles, we are enabled to solve all the cases of rectilineal trigonometry.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

XLVIII. Let A be the right angle of the proposed right-angled triangle, B and C the other two angles; let *a* be the hypotenuse, *b* the side opposite the angle B, *c* the side opposite the angle C. Here we must consider that the two angles A and
b A are complements of each other; and that consequently, according to the different cases, we are entitled to assume $\sin C=\cos B$, $\sin B=\cos C$, and likewise $\operatorname{tang} B=\cot C$, $\operatorname{tang} C=\cot B$. This being fixed, the different problems concerning right-angled triangles are all reducible to the four following cases:

FIRST CASE.

XLIX. *Given the hypotenuse* a, *and a side* b, *to find the other side and the acute angles.*

For determining the angle B, we have (Art. 43.) the proportion $a : b :: R : \sin B$. Knowing the angle, we shall also know its complement $100°-B=C$; we might also find C directly by the proportion $a : b :: R : \cos C$.

As to the third side *c*, it may be found in two ways. Having found the angle B, we can either (Art. 43.) form the proportion

R : *cot* B : : b : c; or the value of c may be obtained directly from the equation $c^2=a^2-b^2$, which gives $c=\surd(a^2-b^2)$, and consequently

$$\log c=\tfrac{1}{2}\log(a+b)+\tfrac{1}{2}\log(a-b).$$

SECOND CASE.

L. *Given the two sides* b *and* c *of the right angle, to find the hypotenuse* a, *and the angles.*

We shall have the angle B (Art. 43.) from the proportion $c:b::\mathrm{R}:tang\,\mathrm{B}$. Next we shall have $\mathrm{C}=100^\circ-\mathrm{B}$. We might also find C directly by the proportion $b:c::\mathrm{R}:tang\,\mathrm{C}$.

Knowing the angle B, we shall find the hypotenuse by the proportion $sin\,\mathrm{B}:\mathrm{R}::b:a$; or a may be obtained directly from the equation $a=\surd(b^2+c^2)$; but as b^2+c^2 cannot be decomposed into factors, this expression is incommodious in calculating with logarithms.

THIRD CASE.

LI. *Given the hypotenuse* a *and an angle* B, *to find the other two sides* b *and* c.

Make the proportions $\mathrm{R}:sin\,\mathrm{B}::a:b$, $\mathrm{R}:cos\,\mathrm{B}::a:c$; they will give the values of b and c. As to the angle C, it is equal to the complement of B.

FOURTH CASE.

LII. *Given a side* b *of the right angle with one of the acute angles, to find the hypotenuse and the other side.*

Knowing one of the acute angles, we shall likewise know the other; hence we may look upon the side b and the opposite angle B as given. To determine a and c, we shall then have the proportions

$$sin\,\mathrm{B}:\mathrm{R}::b:a,\quad \mathrm{R}:cot\,\mathrm{B}::b:c.$$

SOLUTION OF RECTILINEAL TRIANGLES IN GENERAL.

Let A, B, C be the three angles of a proposed rectilineal triangle; a, b, c the sides which are respectively opposite them: the different problems which may occur in determining three of these quantities by means of other three, will all be reducible to the four following cases.

FIRST CASE.

LIII. *Given the side* a *and two angles of the triangle, to find the two other sides* b *and* c.

Two of the angles being known will give us the third; then the two sides b and c will result from the proportions (Art. 44.)

$$sin\ A : sin\ B :: a : b,$$
$$sin\ A : sin\ C :: a : c.$$

SECOND CASE.

LIV. *Given the two sides* a *and* b, *with the angle* A *opposite to one of them, to find the third side* c *and the other two angles* B *and* C.

The angle B may be had by the proportion

$$a : b :: sin\ A : sin\ B.$$

Let M be the acute angle whose sine is $\frac{b\ sin\ A}{a}$; from the value of sin B, we may either take $B=M$, or $B=200°-M$. This ambiguous solution will not occur, however, except we have at once A an acute angle and $b>a$. If the angle A is obtuse, B cannot be so; hence we shall have but one solution; and if, A being acute, we have $b>a$, there will equally be only one solution, because in that case we shall have $M<A$, and by making $B=200°-M$, we should find $A+B>200°$; which it cannot be.

Knowing the angles A and B, we shall also know the third angle C. Then we shall obtain the third side c by the proportion

$$sin\ A : sin\ C :: a : c.$$

We might also deduce c directly from the equation $\frac{cos\ A}{R}=\frac{b^2+c^2-a^2}{2bc}$, which gives $c=\frac{b\,cos\,a}{R}\pm\sqrt{\left(a^2-\frac{b^2\ sin^2\ A}{R^2}\right)}$. But this value will not admit of being computed by logarithms, except by help of an auxiliary angle M or B, which brings it back to the foregoing solution.

THIRD CASE.

LV. *Given two sides* a *and* b, *with their included angle* C, *to find the other two angles* A *and* B *with the third c.*

Knowing the angle C, we shall likewise know the sum of the other two angles $A+B=200°-C$, and their half-sum $\frac{1}{2}(A+B)=100°-\frac{1}{2}C$. Next we shall compute the half-difference of these two angles by the proportion (Art. 42.)

$$a+b : a-b :: tang\ \tfrac{1}{2}(A+B) \text{ or } cot\ \tfrac{1}{2}C : tang\ \tfrac{1}{2}(A-B),$$

in which we consider $a>b$, and consequently $A>B$.

Having found the half-difference, by adding it to the half-sum $\frac{1}{2}(A+B)$, we shall have the greater angle A; by subtracting it from the half-sum, we shall have the smaller angle B. For, A and B being any two quantities, we have always

$$A = \tfrac{1}{2}(A+B)+\tfrac{1}{2}(A+B),$$
$$B = \tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B).$$

Knowing the angles A and B, to find the third side c, we have the proportion

$$\sin A : \sin C :: a : c.$$

* LVI. In trigonometrical calculations, it often happens that two sides a and b are known by their logarithms; in that case, to avoid the trouble of seeking the numbers which correspond to them, we need only seek the angle φ by the proportion $b : a :: R : \varphi \text{ tang } \varphi$. The angle φ will be greater than 50°, since we supposed $a > b$; subtract φ from 50° therefore, and form the proportion $R : \text{tang}(\varphi - 50°) :: \cos \frac{1}{2} C : \text{tang} \frac{1}{2}(A-B)$; from which, as formerly, the value of $\frac{1}{2}(A-B)$ may be determined, and afterwards that of the two angles A and B.

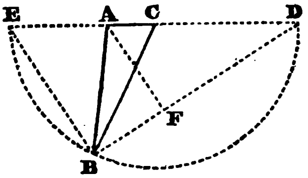
This solution is founded on the property, that, $\text{tang}(50° - \varphi) = \frac{R^2 \text{tang}\,\varphi - R^2 \text{tang}\,50°}{R^2 + \text{tang}\,\varphi\, \text{tang}\,50°}$; now $\text{tang}\,\varphi = \frac{aR}{b}$, and $\text{tang}\,50° = R$; hence $\text{tang}(\varphi - 50° = \frac{R(a-b)}{a+b}$; hence $a+b : a-b :: R : \text{tang}(\varphi - 50°) :: \cot \frac{1}{2} C : \text{tang} \frac{1}{2}(A-B)$.

As for the third side c, it may be found directly by means of the equation $\frac{\cos C}{R} = \frac{a^2+b^2-c^2}{2ab}$, which gives

$$c = \sqrt{\left(a^2 +^2 b^2 - \frac{2ab \cos c}{R}\right)}.$$

But this value is inconvenient for calculating with logarithms, unless the numbers which represent a, b, and $\cos C$ are very simple.

We may observe that the value of c might also be put under these two forms: $c = \sqrt{\left[(a-b)^2 = 4ab \frac{\sin^2 \frac{1}{2} C}{R^2}\right]} = \sqrt{\left[(a+b)^2 \frac{\sin^2 \frac{1}{2} C}{R^2} + a - b^2 \frac{\cos^2 \frac{1}{2} C}{R^2}\right]}$, which is easily verified by means of the formulas $\sin^2 \frac{1}{2} C = \frac{1}{2} R^2 \; \frac{1}{2} R \cos C$, $\cos^2 \frac{1}{2} C = \frac{1}{2} R^2 + \frac{1}{2} R \cos C$. These values will especially be useful, if it is required to compute c with great precision, the angle C and the line $a-b$ being at the same time very small. The latter value shews that c might be the hypotenuse of a right-angled triangle, formed with the sides $(a+b)\frac{\sin \frac{1}{2} \mathrm{C}}{\mathrm{R}}$ and $a-b\frac{\cos \frac{1}{2} \mathrm{C}}{\mathrm{R}}$; a truth of which we may convince ourselves by this very simple construction.



Let CAB be the proposed triangle, in which are known the two sides $\mathrm{CB}=a$, $\mathrm{C\,A}=b$, and the included angle C. From the point C as a centre, with the radius CB equal to the greater of the two given sides, describe a circle meeting the side CA produce in D and E; join BD, BE; and draw AF perpendicular to BD. The angle DBE inscribed in the semicircle is a right-angle; hence the lines AF, BE, are parallel, and we have the proportion BF : AE : : DF : : AD : : *cos* D : R. In the right-angled triangle DEF, we shall in like manner have AF : DA : : *sin* D : R. And substituting the values $\mathrm{DA}=\mathrm{DC}+\mathrm{CA}=a+b$, $\mathrm{AE}=\mathrm{CE}-\mathrm{CA}=a-b$, $\mathrm{D}=\frac{1}{2}\mathrm{C}$, we shall have

$$\mathrm{AF}=\frac{(a\times b)\sin\frac{1}{2}\mathrm{C}}{\mathrm{R}},\ \mathrm{BF}=\frac{(a-b)\cos\frac{1}{2}\mathrm{C}}{\mathrm{R}}.$$

Hence AB the third side of the proposed triangle is actually the hypotenuse of the right-angled triangle ABF, the sides of which are $(a\times b)\frac{\sin\frac{1}{2}\mathrm{C}}{\mathrm{R}}$ and $(a-b)\frac{\cos\frac{1}{2}\mathrm{C}}{\mathrm{R}}$. If in this same triangle, we find the angle ABF opposite the side AF, and subtract from it the angle $\mathrm{CBD}=\frac{1}{2}\mathrm{C}$, we shall have the angle B of the triangle ABC. From which it appears that the solution of the triangle ABC, wherein are known the two sides a and b and the included angle C, is immediately reducible to that of the right-angled triangle ABF, wherein are known the two sides containing the right-angle, namely, $\mathrm{AF}=(a+b)\frac{\sin\frac{1}{2}\mathrm{C}}{\mathrm{R}}$, and $\mathrm{BF}=(a-b)\frac{\cos\frac{1}{2}\mathrm{C}}{\mathrm{R}}$. This construction might, therefore, supply the place of Art. 47.

FOURTH CASE.

LVII. *Given the three sides* a, b, c, *to find the three angles* A, B, C.

The angle A opposite to the side a is found by the formula $\cos\mathrm{A}=\mathrm{R}.\frac{b^2+c^2-a^2}{2\,bc}$; and the other two angles may be determined in the same way. But a different solution may be obtained by a formula more commodious for computing with logarithms.

Recurring to the formula $R^2 - R \cos A = 2 \sin^2 \frac{1}{2} A$, and substituting in it the value of $\cos A$, we shall have $2 \sin^2 \frac{1}{2} A = R^2 . \frac{a^2 - b^2 - c^2 + 2bc}{2bc} = R^2 . \frac{a^2 - (b-c)^2}{2bc} =$ $R^2 . \frac{(a+b-c)\ (a-b+c)}{2bc}$. Hence

$$\sin \tfrac{1}{2} A = R \sqrt{\left(\frac{(a+b-c)\ (a-b+e)}{4bc}\right)}.$$

For the sake of brevity, put $\frac{1}{2}(a+b+c) = p$, or $a+b+c = 2p$; we shall have $a+b-c = 2p - 2c$, $a-b+c = 2p - 2b$; hence

$$\sin \tfrac{1}{2} A = R \sqrt{.} \left(\frac{(p-b)\ (p-c)}{bc}\right).$$

A formula which also gives the proportion

$$bc : (p-b)\ (p-c) :: R^2 : \sin^2 \tfrac{1}{2} A,$$

and which it is easy to calculate by logarithms. Knowing the logarithms of $\sin \frac{1}{2} A$, we shall likewise know $\frac{1}{2} A$, the double of which will be the angle sought.

There are other formulas equally proper for solving the question. Thus, first, the formula $R^2 + R \cos A = 2 \cos^2 \frac{1}{2} A$ gives $\cos^2 \frac{1}{2} A = R^2 . \frac{b^2 + c^2 + 2bc - a^2}{4bc} = R^2 . \frac{(b+)^2 - a^2}{4bc} =$ $R^2 . \frac{(b+c-a)\ (b+c+a)}{4bc}$. But still making $a+b+c = 2p$, we have $b+c-a = 2p - 2a$; hence

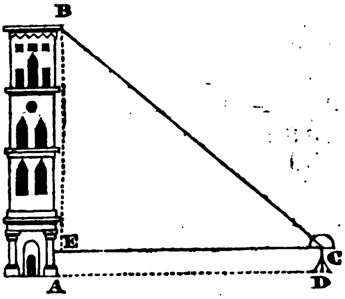
$$\cos \tfrac{1}{2} A = R \sqrt{\left(\frac{(p-a)\, p}{bc}\right)}.$$

And this value being afterwards combined with $\sin \frac{1}{2} A$ will give another formula; for having $\mathit{tang}\ \frac{1}{2} A = \frac{R \sin \frac{1}{2} A}{\cos \frac{1}{2} A}$, we obtain from it

$$\mathit{tang}\ \tfrac{1}{2} A = R \sqrt{\left(\frac{p-b\,.\,p-c}{p\,.\,p-a}\right)}.$$

Examples of the Solution of Rectilineal Triangles.

LVIII. *Example* 1. Suppose the height of a building AB were required, the foot of it being inaccessible.



On the ground which we suppose to be horizontal or very nearly so, measure a base AD, neither very great nor very small in comparison with the altitude AB; then at D place the foot of the circle, or whatever be the instrument, with which we are to measure the angle BCE formed by the horizontal line CE parallel to AD, and by the visual ray directed to the summit of the building. Suppose we find AD or CE = 67. 84 yards, and the angle BCE = 45° 64′: in order to find BE, we shall have to solve the right-angled triangle BCE, in which the angle C and the adjacent side CE are known. Here, according to Case 4., we shall form the proportion R : *tang* 45° 64′ : : 67 . 84 : BE.

Log. tang 45° 64′ . . . . . 9 . 9403263
Log. 67 . 84 . . . . . . 1 . 8314858

Sum — *log*. R . . . . 1 . 7718121.

This logarithm corresponds to 59 . 130; hence we have BE = 59 . 13 yards. To BE add the height of the instrument, which I suppose to be 1 . 12 yds., we shall have the required height AB = 60 . 25 yds.

If, in the same triangle BCE we would know the hypotenuse, form the proportion *cos* 45° 64′ : R : : 67 . 84 : BC.

Log. R. + *Log* 67 . 84 . . . 11 . 8314858
Log. cos 45° 64′ . . . . 9 . 8772784

Difference . . . . 1 . 9542074 = *Log*.BC.

Note. If only the summit B of the building or place whose height is required were visible, we should determine the distance BC by the method shewn in the following example; this distance and the given angle BCE are sufficient for solving the right-angled triangle BCE, whose side increased by the height of the instrument will be the height required.

AMB; those of the arc AMC are the only ones from which AC can be seen under an angle equal to AMC: hence the point M, where those arcs intersect, is likewise the only point from which AB and AC can at once be seen under the angles AMB, AMC. We are now to calculate the position of the point M trigonometrically, from this construction.

Suppose the given quantities, AB = 2500 yards, AC = 7000 yds., BC = 9000 yds., AMB = 30° 80′, AMC = 121° 40′. In the triangle ABC, whose three sides are known, we shall find the angle BAC (Art. 57.) by the formula $\sin^2 \tfrac{1}{2} A = R^2 \cdot \frac{6750 \times 2250}{2500 \times 7000}$; from which we obtain $2 \log \sin \tfrac{1}{2} A = 19.9384483$, $\log \sin \tfrac{1}{2} A = 9.9692241$, $\tfrac{1}{2} A = 76^\circ\ 31'.5$, and, finally, $A = 152^\circ\ 63'$. Draw the diameter AD, and join DB; in the triangle BAD, which is right-angled at B, we shall have the side BA = 2500, and the opposite angle BDA = BMA = 30° 80′; whence results the hypotenuse $AD = \frac{BA \times R}{\sin BDA} = 5374.6$ yds. By drawing the diameter AE, in like manner, and joining CE, we shall have ACE a right-angled triangle, in which are known the side AC = 7000, and the adjacent angle CAE = AMC — 100° = 21° 40′; whence, we shall conclude, that $AE = \frac{R \times AC}{\cos CAE} = 7415$ yds.

Now, if MD and ME are drawn, the two angles AMD, AME being right, the line DME will be straight. It remains then to resolve the triangle DAE in which the line AM, whose magnitude and position we are required to determine, is perpendicular to DE. Now, in this triangle, we have the given sides AD = 5374.6, AE = 7415, and the included angle DAE = BAC + CAE — DAB = 104° 83′. Hence we shall obtain the angle ADE = 50° 93′; and, finally, by the right-angled DAM, we shall have AM = 4190.83 yds. This distance and the angle BAM = 112° 27′, completely determine the position of the point M.*

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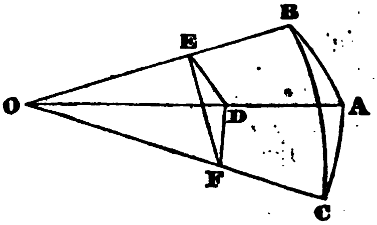
* In order to compute these examples by means of the common tables, we have only to change the angles from the centesimal into the nonagesimal scale; in other respects, the calculation will be exactly the same. The easiest method of effecting this change, in other words, of diminishing the given angles by one tenth of their amount, is to subtract from each, expressed in degrees and decimals of a degree, the same expression having the decimal point a place farther to the left. Thus the angle BAM = 112° 27′ = 112°.27 (*centes.*) = 112°.27 — 11°.227 = 101°.043 = 101° 2′ 34″ 8. (*nonages.*) Ed.

PRINCIPLES FOR THE SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

LXII. *In every right-angled spherical triangle, radius is to the sine of the hypotenuse, as the sine of one of the oblique angles is to the sine of the opposite side.*



Let ABC be the proposed spherical triangle; A its right angle; B and C the other two angles, which we shall call *oblique*, although one or both of them may be right: we shall have the proportion R : *sin* BC : : *sin* B : *sin* AC.

From O, the centre of the sphere, draw the radii OA, OB, OC; then take OF equal to radius in the tables, and from the point F draw FD perpendicular to OA: the line FD will be perpendicular to the plane OAB, because the angle A being right by hypothesis, the two planes OAB, OAC are thus perpendicular to each other. From the point D, draw DE perpendicular to OB; and join EF: the line EF will also be perpendicular to OB, and thus the angle DEF will measure the inclination of the two planes OBA, OBC, and be equal to the angle B of the triangle ABC.

This being proved, in the triangle DEF, right-angled at D, we have R : *sin* DEF : : EF : DF; now the angle DEF $=$ B; and since OF $=$ R, we have EF $=$ *sin* EOF $=$ *sin* BC, DF $=$ *sin* AC. Hence R : *sin* B : : *sin* BC : *sin* AC, or

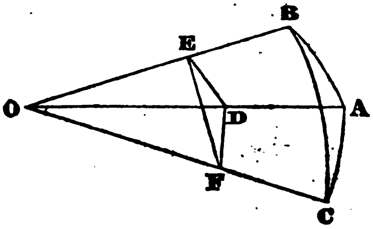
R : *sin* BC : : *sin* B : *sin* AC.

If we designate by a the hypotenuse or side opposite the right angle A, by b the side opposite the angle B, by c the side opposite the angle C, we shall thus have

R : *sin* a : : *sin* B : *sin* b : : *sin* C : *sin* c;

a formula, which of itself furnishes two equations among the parts of the right-angled spherical triangles.

LXIII. *In every right-angled spherical triangle, radius is to the cosine of an oblique angle, as the tangent of the hypotenuse is to the tangent of the side adjacent to that angle.*



Let ABC again be the proposed triangle right-angled at A; we are to shew that R:*cos* B :: *tang* BC : *tang* AB.

For, performing the same construction as before, the right-angled triangle DEF gives the proportion R : *cos* DEF : : EF : ED. But we have DEF = B, EF = *sin* BC, OE = *cos* BC; and in the triangle OED, right-angled at E, we have DE = $\frac{\text{OE}\,\mathit{tang}\,\text{DOE}}{\text{R}}=\frac{\mathit{cos}\,\text{BC}\,\mathit{tang}\,\text{AB}}{\text{R}}$; hence R : *cos* B : : *sin* BC : $\frac{\mathit{cos}\,\text{BC}\,\mathit{tang}\,\text{AB}}{\text{R}}$: : $\frac{\text{R}\,\mathit{sin}\,\text{BC}}{\mathit{cos}\,\text{BC}}$: : *tang* AB; or finally,

R : *cos* B : : *tang* BC : *tang* AB.

Making, as above, BC $= a$ and AB $= c$, we shall have R : *cos* B : : *tang* a : *tang* c, or *cos* B$=\frac{\text{R}\,\mathit{tang}\,c}{\mathit{tang}\,a}=\frac{\mathit{tang}\,c\,\mathit{cot}\,a}{\text{R}}$. The same principle, applied to the angle C, will give *cos* C = $\frac{\text{R}\,\mathit{tang}\,b}{\mathit{tang}\,a}=\frac{\mathit{tang}\,b\,\mathit{cot}\,a}{\text{R}}$.

LXIV. *In every right-angled spherical triangle, radius is to the cosine of a side containing the right angle, as the cosine of the other side is to the cosine of the hypotenuse.*

Let ABC (see the preceding figure) be the proposed triangle right-angled at A; we are to shew that R : *cos* AB : : *cos* AC : *cos* BC.

For, the same construction remaining, the triangle ODF, which is right-angled at D and has the hypotenuse OF = R, will give OD=*cos* DOF=*cos* AC : also the triangle ODE, right-angled at E, will give OE = $\frac{\text{OD}\,\mathit{cos}\,\text{DOE}}{\text{R}}=\frac{\mathit{cos}\,\text{AC}\,\mathit{cos}\,\text{AB}}{\text{R}}$. But in the right-angled triangle OEF, we have OE = *cos* BC;

hence $\cos \mathrm{BC} = \frac{\cos \mathrm{AC} \cos \mathrm{BC}}{\mathrm{R}}$, or what amounts to the same,

$$\mathrm{R} : \cos \mathrm{AC} :: \cos \mathrm{AB} : \cos \mathrm{BC}.$$

This third principle is expressed by the equation $\mathrm{R} \cos a = \cos b \cos c$; it cannot form a second equation, like the two preceding principles, because a change of b for c in it would produce no alteration.

LXV. By means of these three general principles, three others may be found, which are requisite for the solution of right-angled spherical triangles. They might be demonstrated directly, each by a particular construction; but it seems preferable to deduce them, by way of analysis, from the three which are already proved. We shall now do so.

The equations $\sin \mathrm{B} = \frac{\mathrm{R} \sin b}{\sin a}$, $\cos \mathrm{C} = \frac{\mathrm{R}\, \mathrm{tang}\, b}{\mathrm{tang}\, a}$ give, by their division, $\frac{\cos \mathrm{C}}{\sin \mathrm{B}} = \frac{\mathrm{tang}\, b}{\sin b} \cdot \frac{\sin a}{\mathrm{tang}\, a} = \frac{\cos a}{\sin b}$ equal, by the third principle, to $\frac{\cos c}{\mathrm{R}}$. Hence we have this fourth principle,

$$\sin \mathrm{B} : \cos \mathrm{C} :: \mathrm{R} : \cos \mathrm{C},$$

from which also, by changing the letters, there results, $\sin \mathrm{C} : \cos \mathrm{B} :: \mathrm{R} : \cos b$.

The first and the second principle give $\sin \mathrm{B} = \frac{\mathrm{R} \sin b}{\sin a}$, $\cos \mathrm{B} = \frac{\mathrm{R}\, \mathrm{tang}\, c}{\mathrm{tang}\, a}$; hence we deduce $\frac{\sin \mathrm{B}}{\cos \mathrm{B}}$ or $\frac{\mathrm{tang}\, \mathrm{B}}{\mathrm{R}} = \frac{\sin b\, \mathrm{tang}\, a}{\sin a\, \mathrm{tang}\, c} = \frac{\mathrm{R} \sin b}{\cos a\, \mathrm{tang}\, c}$ equal, by the third principle, to $\frac{\mathrm{R}^2 \sin b}{\cos b \cos c\, \mathrm{tang}\, c} = \frac{\mathrm{tang}\, b}{\sin c}$. Hence for a fifth principle, we have the equation $\mathrm{tang}\, \mathrm{B} = \frac{\mathrm{R}\, \mathrm{tang}\, b}{\sin c}$, or the analogy

$$\mathrm{R} : \mathrm{tang}\, \mathrm{B} :: \sin c : \mathrm{tang}\, b;$$

from which also, by changing the letters, there results

$$\mathrm{R} : \mathrm{tang}\, \mathrm{C} :: \sin b : \mathrm{tang}\, c;$$

Lastly, these two formulas give $\mathrm{tang}\, \mathrm{B}\, \mathrm{tang}\, \mathrm{C} = \frac{\mathrm{R}^2\, \mathrm{tang}\, b\, \mathrm{tang}\, c}{\sin b \sin c} =$

$\frac{R^4}{cos\ b\ cos\ c}$ equal, by the third principle, to $\frac{R^3}{cos\ a}$. Hence $R^3 = cos\ a\ tang\ B\ tang\ C$, or $cot\ B\ cot\ C = R\ cos\ a$; or

$$tang\ B : cot\ C :: R : cos\ a.$$

This is the sixth and last principle: it cannot furnish another equation, because the change of B for C in it produces no alteration.

We subjoin a recapitulation of these six principles, whereof four give each two equations:—

I. $R\ sin\ b = sin\ a\ sin\ B$, $R\ sin\ c = sin\ a\ sin\ C$.
II. $R\ tang\ b = tang\ a\ cos\ C$, $R\ tang\ c = tang\ a\ cos\ B$.
III. $R\ cos\ a = cos\ b\ cos\ c$,
IV. $R\ cos\ B = sin\ C\ cos\ b$, $R\ cos\ C = sin\ B\ cos\ c$.
V. $R\ tang\ b = sin\ c\ tang\ B$, $R\ tang\ c = sin\ b\ tang\ C$.
VI. $R\ cos\ a = cot\ B\ cot\ C$.

From these are obtained ten equations including all the relations that can exist between three of the five elements B, C, a, b, c; so that two of these quantities with the right angle being given, the third will immediately be discovered in the form of its sine, cosine, tangent, or cotangent.

LXVI. It is to be observed, that when any element is discovered in the form of its sine only, there will be two values for this element, and consequently two triangles that will satisfy the question; because, the same sine which corresponds to an angle or an arc, corresponds likewise to its supplement. This will not take place, when the unknown quantity is determined by means of its cosine, its tangent, or cotangent. In all these cases, the sign will enable us to decide whether the element in question is less or greater than 100°; the element will be less than 100°, if its cosine, tangent, or cotangent has the sign $+$; it will be greater if one of these quantities has the sign $-$. On this point, likewise, some general principles might be established, which would merely be consequences of the six equations demonstrated above.

From the equation $R\ cos\ a = cos\ b\ cos\ c$, for example, it results, that either the three sides of a right-angled spherical triangle are all less than 100°; or that of those three sides, two are greater than 100°, while the third is less. No other combination can render the sign of $cos\ b\ cos\ c$ like that of $cos\ a$, as the equation requires.

In like manner, the equation $R\ tang\ c = sin\ b\ tang\ C$, in which $sin\ b$ is always positive, proves that $tang\ C$ has always the same sign with *tang c*. Hence *in every right-angled spherical triangle, an oblique angle and the side opposite to it are always of the same species; in other words, are both greater or both less than* 100°.

SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

LXVII. A spherical triangle may have three right angles, and then its three sides are each 100°; it may have only two right angles, in which case, the opposite sides are both 100° each, and there remains an angle with its opposite side, both of which are measured by the same number of degrees. These two kinds of triangles can evidently give rise to no problem; we may, therefore, leave them out of view entirely, and limit our attention to such triangles as have only one right angle.

Let A be the right angle, B and C the other two angles which are called oblique; let *a* be the hypotenuse opposite the angle A; *b* and *c* the sides opposite the angles B and C. Two of the five quanties B, C, *a*, *b*, *c*, being given, the solution of the triangle will always be reducible to one of the six following cases.

FIRST CASE.

LXVIII. *Given the hypotenuse* a, *and a side* b, *the two angles* B *and* C *with the third side* c *may be found by the equations,*

$$\sin \mathrm{B} = \frac{\mathrm{R} \sin b}{\sin a},\ \cos \mathrm{C} = \frac{\operatorname{tang} b \cot a}{\mathrm{R}}\quad \cos c = \frac{\mathrm{R} \cos a}{\cos b}.$$

The angle C and the side *c* have in them no uncertainty as to their signs; the angle B must be of the same species with the side *b*.

SECOND CASE.

LXIX. *Given* b *and* c *the two sides containing the right angle, the hypotenuse* a *and the angles* B *and* C *may be found by the equations,*

$$\cos a = \frac{\cos b \cos c}{\mathrm{R}},\ \operatorname{tang} \mathrm{B} = \frac{\mathrm{R} \operatorname{tang} b}{\sin c},\ \operatorname{tang} \mathrm{C} = \frac{\mathrm{R} \operatorname{tang} c}{\sin b}.$$

There is no ambiguity in any of these values.

THIRD CASE.

LXX. *The hypotenuse* a *and an angle* B *being given, we shall obtain the two sides* b *and* c, *with the other angle* C, *by the equations,*

$$\sin b = \frac{\sin a \sin \mathrm{B}}{\mathrm{R}},\ \operatorname{tang} c = \frac{\operatorname{tang} a \cos \mathrm{B}}{\mathrm{R}},\ \cot \mathrm{C} = \frac{\cos a \operatorname{tang} \mathrm{B}}{\mathrm{R}}.$$

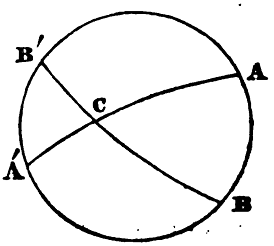
The elements c and C are determined without ambiguity by these formulas; the side b will be of the same species with the angle B.

FOURTH CASE.

LXXI. *Given* b *a side of the right angle with the opposite angle* B, *we shall find the three other elements* a, c, *and* C, *by the formulas,*

$$\sin a = \frac{R \sin b}{\sin B}, \ \sin c = \frac{\tan b \cot B}{R}, \ \sin C = \frac{R \cos B}{\cos b}.$$

In this case, the three unknown elements being determined by means of their sines, the question is susceptible of two solutions. It is evident, accordingly, that the triangle ABC and the triangle AB′C, are both right-angled at A, and have both the same side AC= b, and the same opposite angle B=B′. It only remains for the double values to combine so that c and C shall be of the same species; then the species of c and b will determine that of a, according to the formula $\cos b \cos c = R \cos a$. The value of a may also be derived immediately from the equation $\sin a = \frac{R \sin b}{\sin B}$.



FIFTH CASE.

LXXII. *Given* b *a side of the right angle and the adjacent angle* C, *the other three elements* a, c, B, *may be found by the formulas,*

$$\cot a = \frac{\cot b \cos C}{R}, \ \tan c = \frac{\sin b \tan C}{R}, \ \cos B = \frac{\cos b \sin C}{R}.$$

There is no uncertainty, in this case, with regard to the species of the unknown elements.

SIXTH CASE.

LXXIII. *The oblique angles* B *and* C *being given, the three sides* a, b, c *will result from the formulas,*

$$\cos a = \frac{\cot B \cot C}{R}, \ \cos b = \frac{R \cos B}{\sin C}, \ \cos c = \frac{R \cos C}{\sin B}.$$

In this case, again, there is no ambiguity.

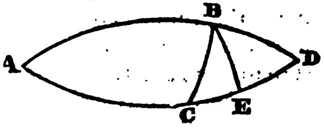
REMARK.

LXXIV. The spherical triangle, whose angles are A, B, C, the sides opposite them being a, b, c, always corresponds to another polar triangle whose angles are supplements of the sides a, b, c, while its sides are supplements of the angles A, B, C; so that, calling the angles of the polar triangle A′, B′, C′, and the sides opposite them a', b', c', we shall have

$$A' = 200^\circ - a,\ B' = 200^\circ - b,\ C' = 200^\circ - c$$
$$a' = 200^\circ - A,\ b' = 200^\circ - B,\ c' = 200^\circ - C$$

This being settled, if a spherical triangle has one of its sides a equal to a quadrant, the corresponding angle A′ of the polar triangle will evidently be right, and thus that triangle will be right-angled. Hence the two data which, in addition to the side of 100°, we must have before solving the proposed triangle, will likewise serve for solving the polar triangle, and consequently for solving the proposed triangle From this property, we derive formulas similar to the foregoing, for the direct solution of spherical triangles which have one side of 100°.

An isosceles triangle divides itself into two right-angled triangles which are equal in all their parts; hence the solution of isosceles spherical triangles likewise depends on that of right-angled spherical triangles.



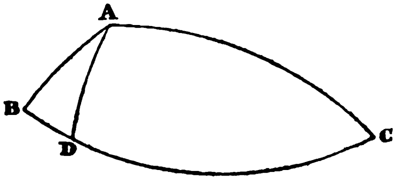
Let ABC be a spherical triangle such that the two sides AB, BC are supplements of each other; the sides AB, AC being produced till they meet, it is evident that BC and BD will be equal, since they are supplements of the same side AB; also it is plain that the parts of the triangle BCD being known, those of the triangle ABC, which is the remainder of the lune AD, are likewise known, and *vice versâ*. Hence the solution of the triangle ABC, whereof two sides together make 200°, is reducible to that of the isosceles triangle BCD, or to that of the right-angled triangle BDE, which is the half of BCD.

When the two sides AB, BC are supplements of each other, the opposite angle ACB, BAC must also be supplements of each other; for BCD is the supplement of BCA, and BCD=D=A. Hence we cannot have $a+c=200^\circ$ without at the same time having $A+C=200^\circ$, which is a reciprocal property.

Thus it appears that the solution of the right-angled spherical triangles includes, *first*, that of spherical triangles having a side equal to a quadrant; *secondly*, that of isosceles spherical triangles; *thirdly*, that of spherical triangles in which the sum of two sides and also of the opposite angle is 200°.

PRINCIPLES FOR THE SOLUTION OF SPHERICAL TRIANGLES IN GENERAL.

LXXV. *In every spherical triangle the sines of the angles are as the sines of the opposite sides.*

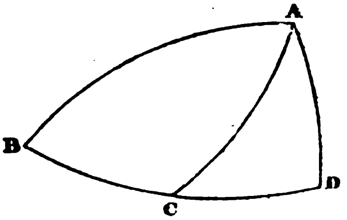


Let AB be any spherical triangle; we are to shew that *sin* B : *sin* C :: *sin* AC : *sin* AB.

From the vertex A, draw AD perpendicular to the opposite side BC; the right-angled triangles ABD, ACD will give the proportions,

$$\sin B : R :: \sin AD : \sin AB$$
$$R : \sin C :: \sin AC : \sin AD$$

Multiplying together the terms of these two proportions, omitting the common factors; we shall have *sin* B : *sin* C :: *sin* AC : *sin* AB.



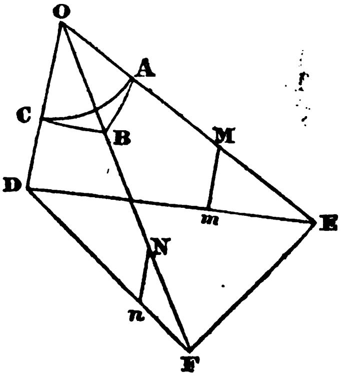
If the perpendicular AD falls without the triangle we shall have the same two proportions, in one of which *sin* C will designate *sin* ACD; but the angles ACD and ACB being supplemental to each other, their sines are equal, and we shall still have *sin* B : *sin* C :: *sin* AC : *sin* AB.

Let *a*, *b*, *c* be the sides respectively equal to the angles A, B, C; by this proposition, we shall have *sin* A : *sin* *a* :: *sin* B : *sin* *b* :: *sin* C : *sin* *c*; which gives the double equation

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

LXXVI. *In every spherical triangle the cosine of an angle is equal to the square of the radius multiplied by the cosine of the opposite side,* minus *the product of the radius by the cosines of the adjacent sides, the whole divided by the product of the sines of those sides: in other words, we shall have, for an angle* C, $\cos C = \frac{R^2 \cos c - R \cos a \cos b}{\sin a \sin b}$; *for the two other angles, we* *shall in like manner have* $\cos B = \frac{R^2 \cos b - R \cos a \cos c}{\sin a \sin c}$,

and $\cos A = \frac{R^2 \cos a - R \cos b \cos c}{\sin b \ \sin c}$.



Let ABC be the proposed triangle, in which BC=a, AC=b, AB=c. From O, the centre of the sphere, draw the indefinite straight lines OA, OB, OC; assume OD at will; and through D draw DE in the plane OCA, and DF in the plane OCB, both perpendicular to OD, and meeting the radii OA, OB produced in E and F; lastly join EF.

The angle D of the triangle EDF is, by construction, the measure of the angle which is formed between the planes OCA, OCB; hence the angle EDF is equal to the angle C of the spherical triangle ACB. Now (Art. 45.) in the triangles DEF, OEF, we have

$$\frac{\cos EDF}{R} = \frac{DE^2 + DF^2 - EF^2}{2 DE . DF},$$

$$\frac{\cos EOF}{R} = \frac{OE^2 + OF^2 - EF^2}{2 OE . OF}.$$

Taking the value of EF^2 in the second, and substituting it in the first, we shall have

$$\frac{\cos EDF}{R} = \frac{DE^2 + DF^2 - OE^2 - OF^2 + 2 OE . OF \frac{\cos EOF}{R}}{2 DE . DF}.$$

But $OE^2 - DE^2 = OD^2$, and $OF^2 - DF^2 = OD^2$, hence we have

$$\cos EDF = \frac{OE.OF.\cos EOF - OD^2.R.}{DE.DF}$$

We have now only to substitute the values which relate to the spherical triangle: but here

$$EDF = C, \ EOF = AB = c, \ \frac{OE}{DE} = \frac{R}{\sin DOE} = \frac{R}{\sin b},$$

$\frac{\mathrm{OF}}{\mathrm{DF}}=\frac{\mathrm{R}}{\sin \mathrm{DOF}}=\frac{\mathrm{R}}{\sin a}$, $\frac{\mathrm{OD}}{\mathrm{DE}}=\frac{\cos \mathrm{DOE}}{\sin \mathrm{DOE}}=\frac{\cos b}{\sin b}$, $\frac{\mathrm{OD}}{\mathrm{DF}}=\frac{\cos \mathrm{DOF}}{\sin \mathrm{DOF}}=\frac{\cos a}{\sin a}$. Hence

$$\cos \mathrm{C}=\frac{\mathrm{R}^2 \cos c-\mathrm{R} \cos a \cos b}{\sin a \sin b}$$

This principle, which, being applied successively to the three angles, affords three equations, is sufficient for solving all the problems of spherical trigonometry: it has the same generality of application in regard to spherical triangles, that Art. 45. has in regard to rectilineal ones. For, since we have always three given elements, by means of which the other three are to be determined, this principle will evidently furnish the equations necessary for solving the problem; equations which it is the province of analysis to develope still farther, in order to deduce from them, according to the different cases, the formulas which are most simple and best adapted for logarithmic calculations.

LXXVII. The principle in question being absolutely general, it must include all the other principles relating to spherical triangles, and particularly the principle explained in Art. 75. Of this it will be easy to satisfy ourselves.

Accordingly, the equation $\cos \mathrm{C}=\frac{\mathrm{R}^2 \cos c-\mathrm{R} \cos a \cos b}{\sin a \sin b}$. gives $\mathrm{R}^2-\cos^2 \mathrm{C}$ or $\sin^2 \mathrm{C}=$

$$\frac{\mathrm{R}^2 \sin^2 a \sin^2 b-\mathrm{R}^2 \cos^2 a \cos^2 b+2\mathrm{R}^3 \cos a \cos b \cos c-\mathrm{R}^4 \cos^2 c}{\sin^2 a \sin^2 b}.$$

Now $\sin^2 a \sin^2 b=(\mathrm{R}^2-\cos^2 a)(\mathrm{R}^2-\cos^2 b)=\mathrm{R}^4-\mathrm{R}^2 \cos^2 a$ $-\mathrm{R}^2 \cos^2 b+\cos^2 a \cos^2 b$. Hence by substituting and extracting the root, we shall have $\sin \mathrm{C}=$

$\frac{\mathrm{R}\sqrt{}}{\sin a \sin b}(\mathrm{R}^4-\mathrm{R}^2 \cos^2 a-\mathrm{R}^2 \cos^2 b-\mathrm{R}^2 \cos^2 c+2\mathrm{R} \cos a \cos b \cos c)$.

For the sake of brevity, put $\mathrm{Z}=$ $\sqrt{}(\mathrm{R}^4-\mathrm{R}^2 \cos^2 a-\mathrm{R}^2 \cos^2 b-\mathrm{R}^2 \cos^2 c+2\mathrm{R} \cos a \cos b \cos c)$, we shall then have

$$\sin \mathrm{C}=\frac{\mathrm{RZ}}{\sin a \sin b}, \text{ or } \frac{\sin \mathrm{C}}{\sin c}=\frac{\mathrm{RZ}}{\sin a \sin b \sin c}.$$

The values of $\cos \mathrm{A}$ and $\cos \mathrm{B}$ would, in like manner, give $\frac{\sin \mathrm{A}}{\sin a}=\frac{\mathrm{RZ}}{\sin a \sin b \sin c}$, $\frac{\sin \mathrm{B}}{\sin b}=\frac{\mathrm{RZ}}{\sin a \sin b \sin c}$; for the quantity Z does not change when a permutation is made between two of the quantities a, b, c; hence we have $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$, which is the principle of Art. 75.

LXXVIII. The values we have just found for $\cos C$ and $\sin C$, may serve for discovering the angles of a spherical triangle, when its three sides are known; though some other formulas are more convenient for logarithmic calculation.

Thus, if in the formula $R^2 - R\cos C = 2\sin^2 \frac{1}{2} C$, the value of $\cos C$ is substituted, we shall have

$$\frac{2\sin^2 \frac{1}{2} C}{R^2} = 1 - \frac{\cos C}{R} = \frac{\cos a \cos b + \sin a \sin b - R \cos c}{\sin a \sin b}.$$

The numerator of this expression is reducible to $R\cos(a-b) - \cos c$; now by the formula (Art. 28) $R\cos q - R\cos p = 2\sin\frac{1}{2}(p+q)\sin\frac{1}{2}(p-q)$, we find $R\cos(a-b) - R\cos c = 2\sin\frac{1}{2}(c-b+a)\sin\frac{1}{2}(c-a+b)$; hence

$$\frac{\sin^2 \frac{1}{2} C}{R} = \frac{\sin\left(\frac{c+b-a}{2}\right)\sin\left(\frac{c+a-b}{2}\right)}{\sin a \sin b},$$

$$\text{or } \sin\tfrac{1}{2} C = R\sqrt{\left\{\frac{\sin\frac{c+b-a}{2}\sin\frac{c+a-b}{2}}{\sin a \sin b}\right\}}.$$

We might evidently obtain similar formulas for expressing $\sin\frac{1}{2} A$ and $\sin\frac{1}{2} B$, by means of the three sides a, b, c.

LXXIX. The general problem of spherical trigonometry consists, as we have already said, in determining three of the six quantities A, B, C, a, b, c, by means of the other three. To effect this, we must have equations among four of those quantities taken in every possible order: now six quantities, when combined four by four, or two by two, give $\frac{6 \times 5}{1 \times 2}$ or 15 combinations; hence there will be fifteen equations to form: though considering only such of the combinations as are essentially different, these fifteen equations are reduced to four. Thus,

1. We have the combination $a\,b\,c\,A$, which, by changing the letters, includes $a\,b\,c\,A$, $a\,b\,c\,B$, $a\,b\,c\,C$.

2. The combination $a\,b\,A\,B$, from which there result $a\,b\,A\,B$, $b\,c\,B\,C$, $a\,c\,A\,C$.

3. The combination $a\,b$ A C, which includes the six $a\,b$ A C, $a\,b$ B C, $a\,c$ A B, $a\,c$ B C, $b\,c$ A B, $b\,c$ A C.

4. Lastly the combination a A B C, which includes the three a A B C, b A B C, c A B C.

In all, therefore, we have fifteen different combinations, but only four of them essentially different.

LXXX. The equation $\cos \mathrm{A} = \dfrac{\mathrm{R}^3 \cos a - \mathrm{R} \cos b \cos c}{\sin b \sin c}$ without any change, represents the first combination $a\,b\,c$ A, and those which depend upon it.

To form the equation corresponding to the combination ab AB, we must eliminate c from the two formulas, which give the values of $\cos$ A and $\cos$ B. This elimination has already been performed (Art. 77.), and the result was $\dfrac{\sin \mathrm{A}}{\sin a} = \dfrac{\sin \mathrm{B}}{\sin b}$.

The third combination is formed of the relation which subsists among a, b, A, C. Here having the two equations

$$\cos \mathrm{A} \sin b \sin c = \mathrm{R}^2 \cos a - \mathrm{R} \cos b \cos c,$$

$$\cos \mathrm{C} \sin b \sin a = \mathrm{R}^2 \cos c - \mathrm{R} \cos b \cos a,$$

we shall first eliminate $\cos c$ from them; which will give $\mathrm{R} \cos \mathrm{A} \sin c + \cos \mathrm{C} \sin a \cos b = \mathrm{R} \cos a \sin b$: then inserting in this the value $\sin c = \dfrac{\sin a \sin \mathrm{C}}{\sin \mathrm{A}}$, we shall have for the third combination,

$$\cot \mathrm{A} \sin \mathrm{C} + \cos \mathrm{C} \cos b = \cot a \sin b.$$

Finally, in order to discover the relation between A, B, C, a, we consider that, in the preceding equation, the term $\cot a \sin b$ $= \mathrm{R} \cos a \dfrac{\sin b}{\sin a} = \mathrm{R} \cos a \dfrac{\sin \mathrm{B}}{\sin \mathrm{A}}$; hence, multiplying this equation by $\sin$ A, we shall have

$$\mathrm{R} \cos \mathrm{A} \sin \mathrm{C} = \mathrm{R} \cos \mathrm{A} \sin \mathrm{B} - \sin \mathrm{A} \cos \mathrm{C} \cos b.$$

If in this equation we mutually change the letters A and B, and also a and b, we shall have

$$\mathrm{R} \cos \mathrm{B} \sin \mathrm{C} = \mathrm{R} \cos b \sin \mathrm{A} - \sin \mathrm{B} \cos \mathrm{C} \cos a.$$

And from these latter two, excluding $\cos b$, we deduce

$$\mathrm{R}^2 \cos \mathrm{A} \sin \mathrm{C} + \mathrm{R} \cos \mathrm{B} \sin \mathrm{C} \cos \mathrm{C} = \cos a \sin \mathrm{B} \sin^2 \mathrm{C}.$$

Hence finally

$$cos\,a=\frac{R^2\,cos\,A+R\,cos\,B\,cos\,C}{sin\,B\,sin\,C};$$

which is the required relation between A, B, C, *a*, or the fourth equation requisite for solving spherical triangles.

LXXXI. This last equation between A, B, C, *a* presents a striking analogy with the first between *a*, *b*, *c*, A; the reason of which must be sought for in the properties of polar or supplemental triangles. We have already seen that the triangle having A, B, C for its angles, and *a*, *b*, *c* for its opposite sides, always corresponds to a polar triangle whose sides are 200°—A, 200°—B, 200°—C, its opposite angles being 200°—*a*, 200°—*b*, 200°—*c*. Now the principle of Art. 76, when applied to this latter triangle, gives

$$cos\,(200°-a)=\frac{R^2 cos\,(200°-A)-R\,cos\,(200°-B)\,cos\,(200°-C)}{sin\,(200°-B)\,sin\,(200°-C)};$$

which may be reduced to

$$cos\,a=\frac{R^2\,cos\,A+R\,cos\,B\,cos\,C}{sin\,B\,sin\,C},$$

as we have found by another method.

This formula immediately solves the case where it is required to determine a side by means of three angles; but, in order to obtain a formula more suitable for logarithmic calculation, we may substitute the value of *cos a* in the equation $1-\frac{cos\,a}{R}=\frac{2\,sin^2\frac{1}{2}a}{R^2}$, which will give us $\frac{sin^2\frac{1}{2}a}{R^2}=$

$$\frac{sin\,B\,sin\,C-cos\,B\,cos\,C-R\,cos\,A}{2\,sin\,B\,sin\,C}=\frac{-R\,cos\,(B+C)-R\,cos\,A}{2\,sin\,B\,sin\,C}.$$

And because (Art. 28.) we have generally $R\,cos\,p+R\,cos\,q=2\,cos\,\frac{1}{2}\,(p+q)cos\,\frac{1}{2}(p-q)$, this equation is reducible to

$$\frac{sin^2\frac{1}{2}a}{R^2}=\frac{-cos\,\frac{1}{2}\,(A+B+C)\,cos\,\frac{1}{2}(B+C-A)}{sin\,B\,sin\,C};$$

where it must be observed that the second member, though under a negative form, is nevertheless always positive. For, generally, we have $sin\,(x-100°)=\frac{sin\,x\,cos\,100°-cos\,x\,sin\,100°}{R}=-cos\,x$; hence

$$-cos\,\tfrac{1}{2}(A+B+C)=sin\left(\frac{A+B+C}{2}-100°\right),$$

a quantity which is always positive, because $A+B+C$ being always included between 200° and 600°, the angle $\frac{1}{2}(A+B+C)-100°$ is included between zero and 200°: likewise $\cos\frac{1}{2}(B+C-A)$ is always positive, because $B+C-A$ cannot exceed 200°; for, in the polar triangle, the side $200°-A$ is less than the sum of the other two $200°-B$, $200°-C$; hence we have $200°-A<400°-B-C$, or $B+C-A<200°$.

Being thus assured that our result will always be positive, for determining a side by means of the angles we shall have the formula

$$\sin\tfrac{1}{2}a=R\sqrt{\left\{\frac{-\cos\frac{A+B+C}{2}\cos\frac{B+C-A}{2}}{\sin B\sin C}\right\}}.$$

LXXXII. Before proceeding farther, it may be observed, that from these general formulas we might deduce the formulas which relate to right-angled spherical triangles. For this purpose, we shall make $A=100°$, both in the four principal formulas and in the formulas derived from them by permutation of the letters. And in the first place, by this substitution, the equation $\cos A\sin b\sin C=R^2\cos a-R\cos b\cos c$ will give

$$R\cos a=\cos b\cos c \qquad (1)$$

The equations derived from this general equation do not contain A, and therefore do not give any new relation in the case of $A=100°$.

The equation $\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}$, in the case of $A=100°$, gives

$$\frac{R}{\sin a}=\frac{\sin B}{\sin b}. \qquad (2)$$

And the derived equation $\frac{\sin A}{\sin a}=\frac{\sin C}{\sin c}$, would, in like manner, give $\frac{R}{\sin a}=\frac{\sin C}{\sin c}$; but this latter is itself derived from the equation (2).

The equation $\cot A\sin C+\cos C\cos b=\cot a\sin b$, in the case of $A=100°$, gives $\cos C\cos b=\cot a\sin b$, or

$$\cos C\tan a=R\tan b \qquad (3)$$

The derived equation $\cot C\sin A+\cos A\cos b=\cot c\sin b$, in the same case, gives $R\cot C=\cot c\sin b$, or

$$R\tan c=\sin b\tan C. \qquad (4)$$

Lastly, the fourth principal equation $\sin B\sin C\cos a=R^2\cos A+R\cos B\cos C$, and its derived equation $\sin A\sin C$

$\cos. b = R^2 \cos B + R \cos A \cos C$, in the case of $A = 100°$, give $\sin B \sin C \cos a = R \cos B \cos C$, and $\sin C \cos b = R \cos B$, or

$$\cot B \cot C = R \cos a, \qquad (5)$$
$$\sin C \cos b = R \cos B. \qquad (6)$$

These are the six equations upon which the solution of spherical triangles depends.

LXXXIII. We shall terminate these principles by demonstrating *Napier's Analogies* which serve to simplify several cases in the solution of spherical triangles.

By combining the values of $\cos A$ and $\cos C$ expressed in terms of a, b, c, we have already (Art. 80.) obtained the equation

$$R \cos A \sin c = R \cos a \sin b - \cos C \sin a \cos b.$$

By a simple permutation, this gives

$$R \cos B \sin c = R \cos b \sin a - \cos C \sin b \cos a.$$

Hence by adding these two equations, and reducing, we shall have

$$\sin c\,(\cos A + \cos B) = (R - \cos C) \sin (a+b).$$

But since $\frac{\sin c}{\sin C} = \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}$, we have

$$\sin C\,(\sin A + \sin B) = \sin C\,(\sin a + \sin b),$$
$$\text{and } \sin C\,(\sin A - \sin B) = \sin C\,(\sin a - \sin b).$$

Dividing these two equations successively by the preceding one; we shall have

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin C}{R - \cos C} \cdot \frac{\sin a + \sin b}{\sin (a+b)}$$
$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin C}{R - \cos C} \cdot \frac{\sin a - \sin b}{\sin (a+b)}$$

And reducing these by the formulas in Articles 29. and 30., there will result

$$\tan g \tfrac{1}{2}(A+B) = \cot \tfrac{1}{2} C . \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}$$
$$\tan g \tfrac{1}{2}(A-B) = \cot \tfrac{1}{2} C . \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}.$$

Hence, two sides a and b with the included angle C being given, the two other angles A and B may be found by the analogies,

$$\cos \tfrac{1}{2}(a+b) : \cos \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2} C : \tan g \tfrac{1}{2}(A+B)$$
$$\sin \tfrac{1}{2}(a+b) : \sin \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2} C : \tan g \tfrac{1}{2}(A-B).$$

If these same analogies are applied to the polar triangle of ABC, we shall have to put $200°-A$, $200°-B$, $200°-a$, $200°-b$, $200°-c$, instead of a, b, A, B, C respectively; and for the result, we shall have these two analogies,

$$\cos\tfrac{1}{2}(A+B) : \cos\tfrac{1}{2}(A-B) :: \operatorname{tang}\tfrac{1}{2}c : \operatorname{tang}\tfrac{1}{2}(a+b)$$

$$\sin\tfrac{1}{2}(A+B) : \sin\tfrac{1}{2}(A-B) :: \operatorname{tang}\tfrac{1}{2}c : \operatorname{tang}\tfrac{1}{2}(a-b),$$

by means of which, when a side c and the two adjacent angles A and B are given, we are enabled to find the two other sides a and b. These four proportions are known by the name of *Napier's Analogies.*

SOLUTION OF SPHERICAL TRIANGLES IN GENERAL.

The solution of spherical triangles includes six general cases, which we shall now explain in succession.

FIRST CASE.

LXXXIV. *The three sides* a, b, c *being given, any angle, for example the angle* A *opposite the side* a, *will be found by the formula:*

$$\sin\tfrac{1}{2}A = R\sqrt{\left\{\frac{\sin\frac{a+b-c}{2} \,.\, \sin\frac{a+c-b}{2}}{\sin b \sin c}\right\}}$$

LXXXV. *Given two sides* a *and* b *with the angle* A *opposite to one of them, to find the third side* c, *and the other two angles* B *and* C.

First. The angle B is found from the equation $\sin B = \dfrac{\sin A \sin b}{\sin a}$.

Secondly. To find the angle C, we must solve the equation,

$$\cot A \sin C + \cos C \cos b = \cot a \sin b.$$

For this purpose, take an auxiliary angle φ, such that we may have $\operatorname{tang}\varphi = \dfrac{\cos b \operatorname{tang} A}{R}$, or $\cot A = \dfrac{\cos b \cos\varphi}{\sin\varphi}$; this value of $\cot A$, being substituted in the equation to be solved, gives $\dfrac{\cos b}{\sin\varphi}(\cos\varphi \sin C + \sin\varphi \cos C) = \cot a \sin b$; whence we obtain

$$\sin(C+\varphi) = \frac{\operatorname{tang} b \sin\varphi}{\operatorname{tang} a}.$$

By this artifice, the two unknown terms of the equation are now reduced to one, from which it is easy to find the angle C.

Thirdly. The side c will be found by the equation

$$\sin c = \frac{\sin a \sin C}{\sin A}.$$

It might also be determined directly, by solving the equation

$$R \cos b \cos c + \cos A \sin b \sin c = R^2 \cos a.$$

For this purpose, put $\cos A \sin b = \frac{R \cos b \sin \varphi}{\cos \varphi}$, or $\tan\!g\, \varphi = \frac{\cos A \,\tan\!g\, b}{R}$; we shall have

$$\frac{\cos b}{\cos \varphi}(\cos c \cos \varphi + \sin c \sin \varphi) = R \cos a.$$

Hence, by first seeking the auxiliary quantity φ from the equation $\tan\!g\, \varphi = \frac{\cos A \,\tan\!g\, b}{R}$, we shall have the side c by the equation

$$\cos(c - \varphi) = \frac{\cos a \cos \varphi}{\cos b}.$$

This second Case may have two solutions, like the analogous Case in rectilineal triangles.

THIRD CASE.

LXXXVI. *Given two sides* a *and* b, *with the included angle* C, *to find the other two angles* A *and* B *and the third side* c.

First. The angles A and B are found by these two equations

$$\cot A = \frac{\cot a \sin b - \cos C \cos b}{\sin C},$$

$$\cot B = \frac{\cot b \sin a - \cos C \cos a}{\sin C};$$

in which the second members may be reduced to a single term, by means of an auxiliary quantity. It is simpler, however, in this case, to make use of Napier's analogies, which give

$$\tan\!g \frac{A-B}{2} = \cot \tfrac{1}{2} C \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)},$$

$$\tan\!g \frac{A+B}{2} = \cot \tfrac{1}{2} C \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}.$$

Secondly. Knowing the angles A and B, the third side c may be computed by the equation $\sin c = \sin a \,.\, \frac{\sin C}{\sin A}$; but for determining c directly, we have the equation

$$R^2 \cos c = \sin a \sin b \cos C + R \cos a \cos b.$$

Assume the auxiliary quantity φ, such that $\sin b \cos C = \cos b \,\mathrm{tang}\, \varphi$, or $\mathrm{tang}\, \varphi = \frac{\cos C \,\mathrm{tang}\, b}{R}$; we shall have

$$\cos c = \frac{\cos b}{\cos \varphi} \cos (a - \varphi)$$

FOURTH CASE.

LXXXVII. *Given two angles* A *and* B *with the adjacent side* c, *to find the other two sides* a *and* b *with the third angle* C.

First. The two sides a and b are given by the formulas

$$\cot a = \frac{\cot A \sin B + \cos B \cos c}{\sin c}$$

$$\cot b = \frac{\cot B \sin A + \cos A \cos c}{\sin c}.$$

They may, however, be computed more easily by Napier's analogies, namely,

$$\sin \frac{A+B}{2} : \sin \frac{A-B}{2} :: \mathrm{tang}\, \tfrac{1}{2} c : \mathrm{tang}\, \frac{a-b}{2},$$

$$\cos \frac{A+B}{2} : \cos \frac{A-B}{2} :: \mathrm{tang}\, \tfrac{1}{2} c : \mathrm{tang}\, \frac{a+b}{2}.$$

Secondly. Knowing a and b, we shall find C by the equation $\sin C = \frac{\sin c \sin A}{\sin a}$; but C may also be found directly, by the equation

$$R^2 \cos C = \cos c \sin A \sin B - R \cos A \cos B.$$

Assume the auxiliary quantity φ, such that

$\cos c \sin B = \cos B \cot \varphi$, or $\cot \varphi = \frac{\cos c \,\mathrm{tang}\, B}{R}$; we shall have

$$\cos C = \cos B \frac{\sin (A - \varphi)}{\sin \varphi}.$$

This case and the preceding one offer no ambiguity.

LXXXVIII. *Given two angles* A *and* B, *with the side* a *opposite one of them, to find the other two sides* b *and* c *and the third angle* C.

First. The side b is found by the equation

$$\sin b = \sin a \cdot \frac{\sin B}{\sin A}.$$

Secondly. The side c depends on the equation

$$\cot a \sin c - \cos B \cos c = \cot A \sin B.$$

Put $\cot a = \cos B \frac{\cos \varphi}{\sin \varphi}$, or $\tan \varphi = \frac{\cos B \tan a}{R}$; we shall have $\frac{\cos B}{\sin \varphi}(\sin c \cos \varphi - \cos c \sin \varphi) = \cot A \sin B$; hence

$$\sin (c - \varphi) = \frac{\tan B \sin \varphi}{\tan A}.$$

Thirdly. The angle C is found by solving the equation

$$\cos a \sin B \sin C - R \cos B \cos C = R^2 \cos A.$$

For this purpose, make $\cos a \sin B = \frac{R \cos B \cos \varphi}{\sin \varphi}$, or $\cot \varphi = \frac{\cos a \tan B}{R}$; we shall have $\frac{\cos B}{\sin \varphi}(\sin C \cos \varphi - \cos C \sin \varphi) = R \cos A$; hence

$$\sin (C - \varphi) = \frac{\cos A \sin \varphi}{\cos B}.$$

This fifth Case, like the second, is susceptible of two solutions, as happens in like manner in the analogous Case of rectilineal triangles.

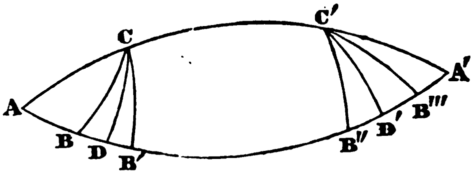
SIXTH CASE.

LXXXIX. *The three angles* A, B, C *being given, we can find any side, for example the side opposite the angle* A, *by the formula*

$$\sin \tfrac{1}{2} a = R \sqrt{\left(\frac{-\cos \frac{1}{2}(A+B+C) \cos \frac{1}{2}(B+C-A)}{\sin B \sin C}\right)}.$$

It may be observed, that of these six general Cases, the last three might have been deduced from the first three, by the property of polar triangles; so that, properly speaking, there are but three different cases in the general solution of spherical triangles. The first Case is solved by a single analogy, as in right-angled triangles; the third is solved in a manner almost equally simple, by means of Napier's analogies. As for the second, it requires two analogies; and also it sometimes admits of two solutions, while the first and third never admit of more than one.

XC. To distinguish whether, in the second Case, for the single given values of A, a, b, there are two triangles which satisfy the question or only one, let us first suppose the angle $A < 100°$,



and let the two sides AC, AB be produced till they meet again in A′. If we take the arc $AC < 100°$, and draw CD perpendicular to AB, the sides AD, CD of the right-angled triangle ACD will be each less than 100°; the line CD will be the shortest distance from the point C to the arc AB; and taking $DB' = DB$, the oblique lines CB′, CB will be equal, and the longer the more they diverge from the perpendicular. Put $AC = b$, $CB = a$; it appears then that a triangle which has $A < 100°$, $b < 100°$, and $a < b$, must necessarily admit of two solutions ACB, ACB′: but if, A and b being still supposed less than 100°, we have $a > b$, in that case the point B′ would pass beyond the point A, and there would be only one solution represented by ABC.

Next suppose $AC' > 100°$: if C′D′ is drawn perpendicular to ABA′, we shall as before have $C'D' < A'C'$; and the arc C′B‴ drawn between D′ and A′ will be greater than C′D′ and less than C′A′: hence making $AC' = b$, $C'B'' = C'B''' = a$ the supposition $A < 100°$ and $b > 100°$ will evidently give two solutions, if $a + b < 200°$, and only one if $a + b > 200°$, because the point B‴ would then pass beyond A′.

By examining upon the same principles the case where the angle A is greater than 100°, the circumstances which determine, if in Case second, the question admits of two solutions or only of one, may be established as follows.

$$A < 100^\circ,\ b < 100^\circ \begin{cases} a > b & \text{one solution.} \\ a < b & \text{two solutions.} \end{cases}$$

$$A < 100^\circ,\ b > 100^\circ \begin{cases} a + b > 200^\circ & \text{one solution.} \\ a + b < 200^\circ & \text{two solutions.} \end{cases}$$

$$A > 100^\circ,\ b < 100^\circ \begin{cases} a + b > 200^\circ & \text{two solutions.} \\ a + b < 200^\circ & \text{one solution.} \end{cases}$$

$$A > 100^\circ,\ b > 100^\circ \begin{cases} a > b & \text{two solutions.} \\ a < b & \text{one solution.} \end{cases}$$

There will only be one solution if $A = 100^\circ$, whether $a = b$, or $a + b = 200^\circ$. There will be two if $b = 100^\circ$.

XCI. These same results may be applied to the fifth Case by means of the polar triangle; and the following circumstances may be deduced from it, to shew whether for given values of A, B, a, there are two triangles which satisfy the question or only one.

$$a > 100^\circ,\ B > 100^\circ \begin{cases} A < B & \text{one solution.} \\ A > B & \text{two solutions.} \end{cases}$$

$$a > 100^\circ,\ B < 100^\circ \begin{cases} A + B < 200^\circ & \text{one solution.} \\ A + B > 200^\circ & \text{two solutions.} \end{cases}$$

$$a < 100^\circ,\ B > 100^\circ \begin{cases} A + B < 200^\circ & \text{two solutions.} \\ A + B > 200^\circ & \text{one solution.} \end{cases}$$

$$a < 100^\circ,\ B < 100^\circ \begin{cases} A < B & \text{two solutions.} \\ A > B & \text{one solution.} \end{cases}$$

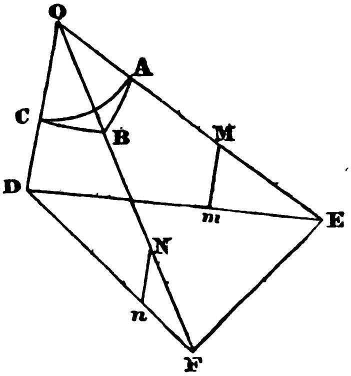
There will only be one solution, if any of the following equalities have place, $a = 100^\circ$, $A = B$, $A + B = 200^\circ$. There will be two, if $B = 100^\circ$.

XCII. In every case, to avoid all useless or false solutions, we must consider

First, that every angle or every side must be less than 200°;

Secondly, that the greater angles lie opposite the greater sides; so that if we have $A > B$, we must likewise have $a > b$, and *vice versâ*.

Examples of the Solution of Spherical Triangles.



XCIII. *Example* 1. Let O, M, N be three points situated in a plane inclined to the horizon: if from these three points, the perpendiculars OD, M *m*, N *n* are drawn to the horizontal plane DEF, the objects situated in O, M, N will be represented on the horizontal plane by their *projections* D, *m*, *n*; and the angle MON by *m* D *n*. This being granted, suppose the angle MON, and the inclinations of its two sides OM, ON to the vertical OD were given; and that we had to find the angle of projection *m* D *n*.

From the point O as a centre and with a radius = 1, describe a spherical surface, meeting the sides OM, ON and the vertical OD, in the points A, B, C; we shall have a spherical triangle ABC whose three sides are known; we shall therefore be able to determine C equal to *m* D *n*, by the formula Case first.

Suppose, for example, the angle MON=AB=64° 44′, 60″, the angle DOM=AC=98° 12′, and the angle DON=BC= 105° 42′; by the formula referred to we shall have

$$\sin^2 \tfrac{1}{2}\,C = R^2 . \frac{\sin 28^\circ\, 57'\, 30''\, \sin 35^\circ\, 87'\, 30''}{\sin 98^\circ\, 12'\, \sin 105^\circ\, 42'} .$$

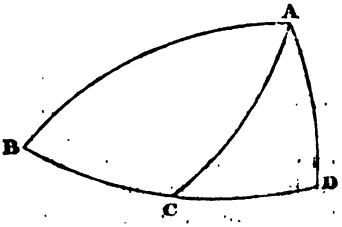
A value which may be computed thus:

L. *sin* 28° 57′ 30″	9.6373956	L. *sin* 98° 12′	9.9998106
L. *sin* 35° 87′ 30″	9.7276562	L. *sin* 105° 42′	9.9984242
Sum + 2 L. R	39.3650518		19.9982348
	19.9982348		
2 L. *sin* ½ C	19.3668170		
L. *sin* ½ C	9.6834085	½ C = 32° 4′ 70″.5	
		C = 64 9 41	

Hence the angle 64° 44′ 60″, measured on a plane inclined to the horizon, is reduced to 64° 9′ 41″ when it is projected on the plane of the horizon.

This problem is useful in the art of taking plans, when the surface to be operated on presents any sensible inequalities, and it is at the same time required to determine the principal positions with great accuracy.

XCIV. *Example* 2. Knowing the latitudes of two points on the globe, and their difference of longitude, to find the shortest distance between them.



Conceive a spherical triangle ACB to be formed by the North Pole C and the two places A and B, whose distance we are required to find. In this triangle we shall know the angle at the Pole ACB, since it is the difference in longitude of the two points A and B; we shall likewise know the two including sides AC, CB, since they are the complements of the latitudes of the points A and B. The third side AB may therefore be determined by the formulas of Case third.

Let A and B, for example, be the Observatories of Paris and Pekin: the north latitude of the one of these places is 54° 26′ 36″, that of the other is 44° 33′ 73″, and their difference in longitude is 126° 80′ 56″. Thus we shall have

$$a = 45^\circ\ 73'\ 64''$$
$$b = 55\ \ 66\ \ 27$$
$$C = 126\ \ 80\ \ 56$$

According to these data, for determining c we shall have the formulas $\tan \varphi = \frac{\cos C \tan b}{R}$, $\cos c = \frac{\cos b \ \cos\ (a - \varphi}{\cos \varphi}$, which are computed thus:

L. *cos* C . . .	9.6114352
L. *tang* *b*. . .	10.0776707
L. *tang* φ. . . .	9.6891059

The angle φ answering in the tables to this logarithmic tangent is 28° 94′ 23.″ We must consider, however, that *cos* C is negative, and that *tang* φ being consequently negative, we must take $\varphi = -28^\circ\ 94'\ 23''$. Having settled this, and observing that $\cos(-\varphi) = \cos \varphi$, we shall finish the calculation thus:

L. *cos* (*a* — φ) . .	9.5880938
L. *cos* φ. . . .	9.8071953
	19.3952891
L. *cos* φ.	9.9554823
L. *cos c*.	9.4418068

Hence the required distance $c = 82° 16' 05''$. This same distance may also be expressed in myriametres by 821 . 605; for a myriametre is the length of an arc measuring ten minutes, a metre being that of an arc measuring one-tenth of a second.

XCV. *Example* 3. To give an example of Case fifth, let us undertake to solve the spherical triangle, wherein are known the two angles $A = 78° 50'$, $B = 54° 0'$, and the side opposite one of them $a = 99° 20' 17''$. By means of this, we find from the Table in Art. 91., that there can be only one solution, since we have at the same time $a < 100°$, $B < 100°$ and $A > B$. This solution is computed as follows:

First. The side b will be found by the formula $\sin b = \sin a \dfrac{\sin B}{\sin A}$.

L. *isn* a	9 . 9999659
L. *sin* B	9 . 8751256
10 — L. *sin* A	0 . 0252525
L. *sin* b	9 . 9003440

Which gives $b = 58° 50' 14''$, or its supplement $141° 49' 86''$; but since the angle B is less than A, the side b must also be less than a; hence the first value is the only proper one.

Secondly. To find the side c we must put $\tan \varphi = \dfrac{\cos B \tan a}{R}$, $\sin (c - \varphi) = \dfrac{\tan B \sin \varphi}{\tan A} = \dfrac{\tan B \cot A \sin \varphi}{R^2}$

		L. *sin* φ...............	9 . 9999220
L. *sin* B............	9 . 8204063	L. *tang* B — LR...	0 . 0547193
L. *tang* a — LR..	1 . 9016731	L. *cot* A...............	9 . 5455236
L. *tang* φ..........	11 . 7220794	L. *sin* (c — φ)....	9 . 6001649
$\varphi = 98° 79' 28''$. 8		$c - \varphi = 26° 7' 70''$. 5.	

Here we have again the choice of taking for $c - \varphi$ the value $26° 7' 70''$. 5, or its supplement $173° 92' 29''$. 5; but by adopting the second value, we should have $c > 100°$; therefore we must keep by the first, which gives $c = 124° 81' 99''$. 3.

Thirdly. In fine, to calculate the angle C directly, we shall take the formulas $\cot \psi = \dfrac{\cos a \tan B}{R}$, $\sin (C - \psi) = \dfrac{\cos A \sin \psi}{\cos B}$.

$$
\begin{array}{ll}
 & \text{L. } \sin \psi \ldots\ldots\ldots\ldots 9 \,.\, 9999563 \\
\text{L. } \cos a \ldots\ldots\ldots\ldots 8 \,.\, 0982928 & \text{L. } \cos \text{A} \ldots\ldots\ldots\ldots 9 \,.\, 5202711 \\
\text{L. } \operatorname{tang} \text{B} - \text{LR}\; 0 \,.\, 0547193 & \text{L. R} - \text{L} \cos \text{B} \ldots 0 \,.\, 1795937 \\
\hline
\text{L. } \cot \psi \ldots\ldots\ldots\ldots 8 \,.\, 1530121 & \text{L. } \sin (\text{C} - \psi)\;\; 9 \,.\, 6998211 \\
\psi = 99^\circ\, 9'\, 45'' \,.\, 5 & \text{C} - \psi = 33^\circ\, 40'\, 54'' \,.\, 5 \\
 & \psi \ldots 99 \;\; 9 \;\; 45 \,.\, 5 \\
\hline
 & \text{C} = 132 \;\; 50 \;\; 0 \,.\, 0
\end{array}
$$

We could not take the supplement of $33^\circ\, 40'\, 54''\,.\,5$ as the value of $C - \psi$; because this would have given us $C > 200$. It is plain, therefore, that the proposed problem admits only of one solution.*

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* Such as wish to know the most useful applications of Trigonometry, cannot do better than consult M. Puissant's *Traité de Topographie, d'Arpentage et de Nivellement* (Treatise on Topography, Surveying and Levelling), Paris, 1807.

END OF THE TRIGONOMETRY.

APPENDIX,

Containing the Solution of various particular Cases in Trigonometry.

XCVI. The manner of solving triangles, which has now been explained, leaves nothing to desire in regard to the universality of its application. There are some cases, however, in which it might be advantageous to employ particular solutions in place of the general ones, either for the purpose of abridging the calculation, or of rendering the results more accurate and more independent of errors in the tables. We shall now exhibit a few of these cases, selecting such as are of most frequent use, or as lead to the most remarkable formulas.

We shall, as formerly, designate the angles of the proposed triangle, spherical or rectilineal, by A, B, C; the sides which lie respectively opposite them by a, b, c. We shall likewise suppose the radius of the tables $= 1$, an hypothesis, which cannot affect the universality of the results. The angles A, B, C are expressed in calculation, either by the degrees or the absolute lengths of the arcs which measure them, these arcs being taken in a circle whose radius is 1. If an angle or arc x is very small, then in place of $\sin x$, and $\cos x$, their values expressed in series may be employed, namely, $\sin x = x - \frac{x^3}{1.2.3} + \&c.$ $\cos x = 1 - \frac{x^2}{1.2} + \&c$; but in that case, x must be expressed in parts of the radius. An arc being found in parts of the radius, to obtain its value in minutes, it must be multiplied by the number of minutes contained in the radius; which number is $\frac{20000}{\pi} = 6366.1977237$, its logarithm being 3.80388012297.

§ I. *Of rectilineal triangles, two of whose angles are very small.*

XCVII. Suppose two angles A and B to be very acute, and consequently C to be very obtuse; we may put $\sin A = A - \frac{1}{6} A^3$, $\sin B = B - \frac{1}{6} B^3$, and $\sin C = \sin (A + B) = A + B - \frac{1}{6} (A + B)^3$. Hence if the side c with the adjacent angles A and B are known, the other two sides may be found by the formulas $a = \frac{c \sin A}{\sin (A + B)}$, $b = \frac{c \sin B}{\sin (A + B}$, which, by substituting the preceding values, and reducing, become

$$a=\frac{c\,A}{A+B}\left(1+\frac{2\,A\,B+B^2}{6}\right)$$

$$b=\frac{c\,B}{A+B}\left(1+\frac{A^2+2\,A\,B}{6}\right),$$

and from this, results $a+b-c=\frac{1}{2}\,c\,AB$. These values are accurate, excepting such terms as contain four dimensions in A and B.

XCVIII. In the second place, suppose that we have the two sides a and b given, with the contained angle $C=\pi-\theta$, θ being very small. We shall first have $c^2=a^2+b^2+2ab\cos\theta=a^2+b^2+2ab\,(1-\frac{1}{2}\theta^2)$ $=(a+b)^2-ab\,\theta^2$; hence

$$c=a+b-\tfrac{1}{2}\cdot\frac{a\,b\,\theta^2}{a+b}.$$

Next, the angle A will be found by the equation $\sin A=\frac{a}{c}\sin C=\frac{a}{c}\sin\theta$, from which, by substituting the value of c, and that of $\sin\theta$, we deduce $\sin A=\frac{a}{a+b}\left(\theta+\frac{1}{2}\cdot\frac{a\,b}{(a+b)^2}\theta^3-\frac{1}{6}\theta^3\right)=$
$\frac{a\,\theta}{a+b}\left(1+\frac{a\,b-a^2-b^2}{(a+b)^2}\cdot\frac{\theta^2}{6}\right)$.
Hence $A=\sin A+\frac{1}{6}\sin^3 A=\frac{a\,\theta}{a+b}+\frac{a\,b\,(a-b)}{(a+b)^2}\cdot\frac{\theta^3}{6}$. Hence also we might deduce the value of B, by interchanging the letters a and b: but A being known, we have immediately $B=\theta-A$. If θ is given in minutes, in order to obtain A likewise expressed in minutes, we must substitute, in the preceding formulas, the ratios $\frac{A}{R}$, $\frac{\theta}{R}$ in place of A and θ, R being the number of minutes contained in the radius. We shall thus have

$$c=a+b-\frac{\frac{1}{2}\,a\,b}{a+b}\cdot\left(\frac{\theta}{R}\right)^2$$

$$A=\frac{a\,\theta}{a+b}\left[1+\frac{b\,(a-b)}{6\,(a+b)^2}\left(\frac{\theta}{R}\right)^2\right]$$

XCIX. To give an example of these formulas, put $a=1000$ yds., $b=2400$ yds., $C=199^\circ\ 32'$ or $\theta=68'$; we shall have $a+b-c=$ $\frac{1200000}{3400}\cdot\left(\frac{68}{R}\right)^2=0.037806$, whence $c=3399.962194$ yds. Then by a first approximation, we have $A=\frac{a\,\theta}{a+b}=20'$, and $B=\theta-A=$

48′, but the whole formula gives $A = 20' \left[1 - \frac{2400 \times 1400}{6\,(3400)^2} \left(\frac{68}{R}\right)^2 \right]$ $= 19'.99988946$, and consequently $B = 48'.00011054$, values which are correct to the last decimal.

§ II. *Solution of the third case of rectilineal triangles, by the method of series.*

C. The two sides a and b, with the included angle C being given, to find the angle B, we have the proportion $b : a :: \sin B : \sin (B + C)$, which gives $a \sin B = b\,(\sin B \cos C + \cos B \sin C)$, and consequently $\frac{\sin B}{\cos B} = \frac{b \sin C}{a - b \cos C}$. If in this equation, instead of the sines and cosines, we insert their values in imaginary exponentials (Art. 35.), we shall have

$$\frac{e^{B\sqrt{-1}} - e^{-B\sqrt{-1}}}{e^{B\sqrt{-1}} + e^{-B\sqrt{-1}}} = \frac{b\left(e^{C\sqrt{-1}} - e^{-\sqrt{-1}}\right)}{2a - b\left(e^{C\sqrt{-1}} + e^{-\sqrt{-1}}\right)}.$$

Whence we obtain

$$e^{2B\sqrt{-1}} = \frac{a - be^{-C\sqrt{-1}}}{a - be^{C\sqrt{-1}}}.$$

Taking the logarithms of each member, and expanding the second into a common series by the formula

$L\,(a - x) = L\,a - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \&c.$, we shall have

$$2B\sqrt{-1} = \frac{b}{a} e^{C\sqrt{-1}} + \frac{b^2}{2a^2} e^{2C\sqrt{-1}} + \frac{b^3}{3a^3} e^{3C\sqrt{-1}} \;\&c.$$

$$- \frac{b}{a} e^{-C\sqrt{-1}} - \frac{b^2}{2a^2} e^{-2C\sqrt{-1}} - \frac{b^3}{3a^3} e^{-3C\sqrt{-1}} - \&c.$$

Hence, dividing by $2\sqrt{-1}$, and observing that $e^{mC\sqrt{-1}} - e^{-mC\sqrt{-1}} = 2\sqrt{-1} \sin m\,C$, we shall have

$$B = \frac{b}{a} \sin C + \frac{b^2}{2a^2} \sin 2C + \frac{b^3}{3a^3} \sin 3C + \frac{b^4}{4a^4} \sin 4C + \&c.$$

This is the value of the angle B, expressed in parts of the radius, by a series, the law of which is very simple, and which will converge the faster, the less b is in proportion to a.

The value just found must likewise satisfy the equation,

$\tan (B + \frac{1}{2}C) = \frac{a+b}{a-b} \tan \frac{1}{2} C$, which is the same as

$tang\ \frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{1}{2}C$, and differs only in form from the equation $\frac{\sin B}{\cos B}=\frac{b\sin C}{a-b\cos C}$.

CI. The angle B being known, we shall have the third angle $A = 200^{\circ} - B - C$. As for the third side c, it depends on the equation $c^2 = a^2 - 2ab\cos c + b^2$, which by extracting the root gives

$$c = a - b\cos C + \frac{b^2}{2a}\sin^2 C + \frac{b^3}{2a^2}\sin^2 C\cos C - \&c.$$

But this series has not a regular movement, and cannot be continued at will. On the other hand, a very simple series may be found for the value of the hyperbolic logarithm of c. Thus, it is manifest that the quantity $a^2 - 2ab\cos C + b^2 = \left(a - be^{C\sqrt{-1}}\right)\left(a - be^{C-\sqrt{-1}}\right)$, for the developed product of these two factors gives

$$a^2 - ab\left(e^{C\sqrt{-1}} + e^{-C\sqrt{-1}}\right) + b^2, \text{ or } a^2 - 2ab\cos C + b^2.$$

Hence we have $c^2 = \left(a - be^{C\sqrt{-1}}\right)\left(a - be^{-C\sqrt{-1}}\right)$.

Taking the logarithm of each member, there will result

$$2\,L\,c = L\,a - \frac{b}{a}e^{C\sqrt{-1}} - \frac{b^2}{2a^2}e^{2C\sqrt{-1}} - \frac{b^3}{3a^3}e^{3C\sqrt{-1}} - \&c.$$

$$+ L\,a - \frac{b}{a}e^{-C\sqrt{-1}} - \frac{b^2}{2a^2}e^{-2C\sqrt{-1}} - \frac{b^3}{3a^3}{}^{-3C\sqrt{-1}} - \&c.$$

Hence, by again reducing, according to the formula $e^{mC\sqrt{-1}} + e^{-mC\sqrt{-1}} = 2\cos mC$, we shall have

$$L\,c = L\,a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{b^3}{3a^3}\cos 3C - \&c.$$

a series not less elegant than the one which gives the value of B. If the logarithms are required to be those of the common tables, we have only to multiply the different terms by the modulus 0·43429448.

§ III. *Solution of the third case of spherical triangles by means of series.*

CII. In the preceding paragraph, we have shewn, that the value of x drawn from the equation $tang\ x = \frac{m+n}{m-n}\,tang\ \frac{1}{2}C$ may be expressed by this series,

$$x\tfrac{1}{2}C + \frac{n}{m}\sin C + \frac{n^2}{2m^2}\sin 2C + \frac{n^3}{3m^3}\sin 3C + \&c.$$

Now, in a spherical triangle, when the two sides a and b, and the included angle C are known, by Napier's Analogies (Art. 86) we have

$$\cot\frac{A-B}{2}=\frac{\sin\left(\frac{1}{2}a+\frac{1}{2}b\right.}{\sin\left(\frac{1}{2}a-\frac{1}{2}b\right)}\tan\frac{1}{2}C$$

$$=\frac{\sin\frac{1}{2}a\cos\frac{1}{2}b+\cos\frac{1}{2}a\sin\frac{1}{2}b}{\sin\frac{1}{2}a\cos\frac{1}{2}b-\cos\frac{1}{2}a\sin\frac{1}{2}b}\tan\frac{1}{2}C$$

$$\cot\frac{A+B}{2}=\frac{\cos\left(\frac{1}{2}a+\frac{1}{2}b\right)}{\cos\left(\frac{1}{2}a-\frac{1}{2}b\right)}\tan\frac{1}{2}C$$

$$=\frac{\cos\frac{1}{2}a\cos\frac{1}{2}b-\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}a\cos\frac{1}{2}b+\sin\frac{1}{2}a\sin\frac{1}{2}b}\tan\frac{1}{2}C$$

Hence, applying the preceding formula, and still supposing $b < a$, we shall have

$$\frac{A-B}{2}=100^{0}-\frac{1}{2}C-\frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a}\sin C-\frac{\tan^{2}\frac{1}{2}b}{2\tan^{2}\frac{1}{2}a}\sin 2C$$
$$-\frac{\tan^{3}\frac{1}{2}b}{3\tan^{3}\frac{1}{2}a}\sin 3C-\&c.$$

$$\frac{A+B}{2}=100^{0}-\frac{1}{2}C+\frac{\tan\frac{1}{2}b}{\cot\frac{1}{2}a}\sin C-\frac{\tan^{2}\frac{1}{2}b}{2\cot^{2}\frac{1}{2}a}\sin 2C$$
$$+\frac{\tan^{3}\frac{1}{2}b}{3\cot^{3}\frac{1}{2}a}\sin 3C-\&c.$$

two series, the law of which is very simple, and which are the more convergent the less b is. The first is always convergent, because we suppose $b < a$; the second will be so likewise, if we have tang $\frac{1}{2}b < \cot\frac{1}{2}a$, or $a+b < 200^{\circ}$. It would be divergent and false, if we had $a+b > 200^{\circ}$; but this case may always be avoided, because the solution of the triangle BCA (see the fig. of Art. 71.) when it has $CA+CB > 200^{\circ}$, may always be reduced to that of the triangle A′CB′, which has $CA'+CB' < 200^{\circ}$. We may also observe, that this second series is most convergent when a and b are both very small; in which case, the third side c will be very small likewise, because we must have $c < a+b$, and the spherical triangle will differ very little from a plane triangle. The excess of the sum of its three angles above two right angles will then be expressed thus:

$$A+B+C-200^{\circ}=\tfrac{2}{1}\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\sin C-\tfrac{2}{2}\tan^{2}\tfrac{1}{2}a\tan^{2}\tfrac{1}{2}b\sin 2C$$
$$+\tfrac{2}{3}\tan^{3}\tfrac{1}{2}a\tan^{3}\tfrac{1}{2}b\sin 3C-\&c.$$

CIII. To find the third side c of the proposed triangle, we have the equation $\cos c=\cos a\cos b+\sin a\sin b\cos C$, from which it is easy to deduce the two following:

$$\sin^{2}\tfrac{1}{2}c=\sin^{2}\tfrac{1}{2}a\cos^{2}\tfrac{1}{2}b-2\sin\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos C+\cos^{2}\tfrac{1}{2}a\sin^{2}\tfrac{1}{2}b$$
$$\cos^{2}\tfrac{1}{2}c=\cos^{2}\tfrac{1}{2}a\cos^{2}\tfrac{1}{2}b+2\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos C+\sin^{2}\tfrac{1}{2}a\sin^{2}\tfrac{1}{2}b.$$

By the form of these values it is manifest, that $\sin\frac{1}{2}c$ may be regarded as the third side of a rectilineal triangle, in which the two sides $\sin\frac{1}{2}a\cos\frac{1}{2}b$ and $\cos\frac{1}{2}a\sin\frac{1}{2}b$, with the included angle C, are known; in like manner, $\cos\frac{1}{2}c$ is the third side of a rectilineal triangle, whose two sides are $\cos\frac{1}{2}a\cos\frac{1}{2}b$ and $\sin\frac{1}{2}a\sin\frac{1}{2}b$, the included angle being

$200^\circ - \mathrm{C}$. Hence by the formula for rectilineal triangles (Art. 101,) we have

$$log\ sin\ \tfrac{1}{2}c = log\ (sin\ \tfrac{1}{2}a\ cos\ \tfrac{1}{2}b) - \frac{tang\ \frac{1}{2}b}{tang\ \frac{1}{2}a}\ cos\ \mathrm{C} - \frac{tang^2\ \frac{1}{2}b}{2 tang^2 \frac{1}{2}a}\ cos 2\mathrm{C} - \&c.$$

$$log\ cos\ \tfrac{1}{2}c = log(cos\ \tfrac{1}{2}a\ cos\ \tfrac{1}{2}b) + \frac{tang\ \frac{1}{2}b}{cot\ \frac{1}{2}a}\ cos\ \mathrm{C} - \frac{tang^2\ \frac{1}{2}b}{2\ cot^2\ \frac{1}{2}a}\ cos\ 2\ \mathrm{C} + \&c.$$

It is farther to be observed, that as each of the rectilineal triangles in question may be solved by means of a right-angled rectilineal triangle, the solution of the proposed spherical triangle may be directly reduced to that of a right-angled rectilineal triangle.

By this means, we find that $sin\ \frac{1}{2}c$ is the hypotenuse of a right-angled triangle, whose sides are $sin\ \frac{1}{2}(a+b)\ sin\ \frac{1}{2}\mathrm{C}$ and $sin\ \frac{1}{2}(a-b)\ cos\ \frac{1}{2}\mathrm{C}$. In like manner, $cos\ \frac{1}{2}c$ is the hypotenuse of a right-angled triangle, whose sides are $cos\ \frac{1}{2}(a-b)\ cos\ \frac{1}{2}\mathrm{C}$, and $cos\ \frac{1}{2}(a+b)\ sin\ \frac{1}{2}\mathrm{C}$.

Also, if the angle lying opposite the side $sin\ \frac{1}{2}(a-b)\ cos\ \frac{1}{2}\mathrm{C}$ in the first triangle is called M, the angle lying opposite the side $cos\ \frac{1}{2}(a-b)\ cos\ \frac{1}{2}\mathrm{C}$ in the second triangle being called N, it follows, from Napier's Analogies, that we shall have $\frac{\mathrm{A}-\mathrm{B}}{2} = \mathrm{M}$, and $\frac{\mathrm{A}+\mathrm{B}}{2} = \mathrm{N}$ or $= 200^\circ - \mathrm{N}$; namely: $\frac{\mathrm{A}+\mathrm{B}}{2} = \mathrm{N}$ if $a+b \angle 200^\circ$, and $\frac{\mathrm{A}+\mathrm{B}}{2} = 200^\circ - \mathrm{N}$ if $a+b > 200^\circ$. Hence, in every spherical triangle where two sides, a and b, with the included angle C, are known, each of the quantities $\frac{1}{2}c$, $\frac{\mathrm{A}+\mathrm{B}}{2}$, $\frac{\mathrm{A}-\mathrm{B}}{2}$ may be found directly, by the solution of a right-angled rectilineal triangle, in which the two sides and the right angle are known.

It follows, likewise, that after having found the angle M, or $\frac{\mathrm{A}-\mathrm{B}}{2}$, by the formula $tang\ \mathrm{M} = \frac{sin\ \frac{1}{2}(a-b)}{sin\ \frac{1}{2}(a+b)}\ cot\ \frac{1}{2}\mathrm{C}$, the third side may be computed by the formula $sin\ \frac{1}{2}c = \frac{sin\ \frac{1}{2}(a-b) cos\ \frac{1}{2}\mathrm{C}}{sin\ \mathrm{M}} = \frac{sin \frac{1}{2}(a+b) sin \frac{1}{2}\mathrm{C}}{cos\ \mathrm{M}}$.

Note. The formulas exhibited in this paragraph will easily apply to the solution of the fifth case of spherical triangles, because this case may be referred to the third, by the property of the polar triangle.

§ IV. *Solution of a spherical triangle, two of whose sides are little different from* 100°.

CIV. Let a and b be the two given sides differing little from 100°; we propose to determine the angle C, by means of the three sides a, b, c.

If the sides a and b were exactly equal to 100°, we should have $\mathrm{C} = c$; therefore, since a and b differ very little from 100°, the angle C will be measured by an arc very little different from c. Put $a = 100^\circ + \alpha$, $b =$

$100^\circ+\beta$, $C=c+x$; substituting these values in the equation $\cos C=\frac{\cos c-\cos a\cos b}{\sin a\sin b}$, we shall have $\cos(c+x)=\frac{\cos c-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$. But since α and β are supposed to be very small, by merely neglecting the terms in which α and β rise to the fourth degree, we may put

$\sin\alpha\sin\beta=\alpha\beta$, $\cos\alpha\cos\beta=1-\frac{\alpha^2}{2}-\frac{\beta^2}{2}$, which will give

$\cos(c+x)=\frac{\cos c-\alpha\beta}{1-\frac{1}{2}\alpha^2-\frac{1}{2}\beta,}=(1+\frac{1}{2}\alpha^2+\frac{1}{2}\beta^2)\cos c-\alpha\beta$. But neglecting the square of x, we have $\cos(c+x)=\cos c-x\sin c$; hence

$$x=\frac{\alpha\beta-\frac{1}{2}(\alpha^2+\beta^2)\cos c}{\sin c}.$$

And since x is of the second order in comparison with α and β, it is plain that no quantities have been neglected in this value, except quantities of the fourth degree. Put $\frac{1}{2}(\alpha+\beta)=p$, $\frac{1}{2}(\alpha-\beta)=q$, or $\alpha=p+q$, $\beta=p-q$; we shall have, under a simpler form,

$$x=p^2\left(\frac{1-\cos c}{\sin c}\right)-q^2\left(\frac{1+\cos c}{\sin c}\right)=p^2\,\text{tang}\,\tfrac{1}{2}c-q^2\cot\tfrac{1}{2}c.$$

This value is expressed in parts of the radius; but as in practice p and q are given in seconds, if we require x likewise to be expressed in seconds, we must put

$$x=\frac{p^2}{R}\,\text{tang}\,\tfrac{1}{2}c-\frac{q^2}{R}\cot\tfrac{1}{2}c,$$

R being the number of seconds contained in the radius, a number whose logarithm $=5.8038801$. Knowing x, we shall have the required angle $C=c+x$.

The formula we have just found is of use, in geodesical operations, for reducing to the horizon such angles as have been observed on inclined planes; it is more expeditious, and requires less extensive tables, than the formulas of Case First in spherical triangles, an example of which was given in Art. 93. Yet if the elevations or depressions α and β were more than 2 or 3 degrees, it would be better to make use of the general method.

§ V. *Solution of spherical triangles, whose sides are very small in comparison with the radius of the sphere.*

CV. When the sides a, b, c, are very small in comparison with the radius of the sphere, the proposed triangle differs little from a rectilineal one; and considering it as such, a first approximate solution of it may be obtained, in which, however, the excess of its three angles above two right angles is neglected. To obtain a more approximate solution, this excess must be taken into account, as may be done very easily by means of a general principle which we shall now demonstrate.

Let r be the radius of the sphere in which the proposed triangle is situated; if a similar triangle is conceived to be traced on the sphere whose radius is 1, the sides of this triangle will be

$\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$, and we shall have $\cos A = \dfrac{\cos\frac{a}{r} - \cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}}$. But since r is very great in comparison with a, b, c, we shall have (Art. 35.) very approximately $\cos\frac{a}{r} = 1 - \frac{a^2}{2r^2} + \frac{a^4}{2.3.4r^4}$, $\cos\frac{b}{r} = 1 - \frac{b^2}{2r^2} + \frac{b^4}{2.3.4r^4}$, $\cos\frac{c}{r} = 1 - \frac{c^2}{2r^2} + \frac{c^4}{2.3.4r^4}$, $\sin\frac{b}{r} = \frac{b}{r} - \frac{b^3}{2.3r^3}$, $\sin\frac{c}{r} = \frac{c}{r} - \frac{c^3}{2.3r^3}$. Substituting these values in the preceding equation, and neglecting the terms of more than four dimensions in a, b, c, we shall have

$$\cos A = \frac{\dfrac{b^2+c^2-a^2}{2r^2} + \dfrac{a^4-b^4-c^4}{24r^4} - \dfrac{b^2c^2}{4r^4}}{\dfrac{bc}{r^2}\left(1 - \dfrac{b^2}{6r^2} - \dfrac{c^2}{6r^2}\right)}$$

Multiplying the two terms of this fraction by $1 + \frac{b^2+c^2}{6r^2}$, after reducing, we shall have

$$\cos A = \frac{b^2+c^2-a^2}{2bc} + \frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2}{24bcr^2}.$$

Now let A′ be the angle opposite the side a, in the rectilineal triangle, whose sides are equal in length to the arcs a, b, c; we shall have $\cos A' = \frac{b^2+c^2-a^2}{2bc}$, and $4b^2c^2\sin^2 A' = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4$. Hence

$$\cos A = \cos A' - \frac{bc}{6r^2}\sin^2 A'.$$

Put $A = A' + x$; rejecting the square of x, we shall have $\cos A = \cos A' - x\sin A'$; whence it appears that $x = \frac{bc}{6r^2}\sin A'$; and x being of the second order in comparison with $\frac{b}{r}$ and $\frac{c}{r}$, this result must be accurate, as far as the quantities of the fourth order. Hence we shall have

$$A = A' + \frac{bc}{6r^2}\sin A'.$$

But $\frac{1}{2}\, b\, c\, \sin A'$ is the area of the rectilineal triangle having a, b, c for its three sides, and does not sensibly differ from that of the proposed spherical triangle. Hence, either of these areas, being designated by α, we shall have $A = A' + \frac{\alpha}{3r^2}$, or $A' = A - \frac{\alpha}{3r^2}$. We may likewise have $B' = B - \frac{\alpha}{3r^2}$, $C' = C - \frac{\alpha}{3r^2}$, and en results $A' + B' + C'$ or $200° = A + B + C - \frac{a}{r^2}$. We may, therefore, consider $\frac{\alpha}{r^2}$ as the excess of the three angles in the proposed spherical triangle above two right angles; and this being granted, we have the following remarkable theorem, which reduces the solution of very small spherical triangles to that of rectilineal triangles.

A spherical triangle, whose sides are very small in comparison with the radius of the sphere, being proposed, if each of its angles be diminished by a third part of the difference between the sum of its three angles and two right angles, the angles so diminished may be taken as the angles of a rectilineal triangle, whose sides are equal in length to those of the proposed spherical triangle; or in other terms:

The very slightly curved spherical triangle, whose angles are A, B, C, *and whose opposite sides are* a, b, c, *always corresponds to a rectilineal triangle, whose sides* a, b, c *are of the same length, and whose opposite angles are* $A - \frac{1}{3}\varepsilon$, $B - \frac{1}{3}\varepsilon$, $C - \frac{1}{3}\varepsilon$, ε *being the excess of the sum of all the angles in the proposed spherical triangle, above two right angles.*

CVI. The excess ε or $\frac{\alpha}{r^2}$, which is proportional to the area of the triangle, may always be calculated *a priori* from the data of the spherical triangle considered as rectilineal. If two sides b, c, and the included angle A are given, we shall have the area $\alpha = \frac{1}{2}\, bc \sin A$; if the side a, and the two adjacent angles B, C are given, we shall have the area $\alpha = \frac{1}{2}\, a^2 \frac{\sin B \sin C}{\sin (B + C)}$. We shall then have $\varepsilon = \frac{\alpha}{r^2} R$, R being the number of seconds included in the radius; and ε will thus be expressed in seconds.

In applying these formulas to triangles, drawn upon the surface of the earth considered as spherical,* the sides a, b, c, and likewise r the radius of the earth, must be considered as expressed in metres.† Now since the fourth part of the meridian $\frac{1}{2}\,\pi\, r$ is equal to 10000000 metres, we find $log\ r = 6.\ 8038801$; on the other hand, the radius R expressed in seconds has 5. 8038801 for its logarithm. Hence, if to the logarithm of α, the area expressed in square metres, we add the constant logarithm 2.196119, and take ten units from the sum, we shall have the logarithm of the excess ε expressed in seconds.

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* In geodesical operations, the triangles are most frequently formed between three stations unequally distant from the centre of the earth: but, by suitable reductions, the observed triangles are changed for those triangles, which result from projecting all the three stations on the same spherical surface, perpendicular to the direction of gravity.

† The *metre*, or base of the new French system of measures, is equal to 39.371 English inches.—ED.

Knowing ε, we subtract $\frac{1}{3}\varepsilon$, or suppose it to be subtracted from each angle of the proposed spherical triangle; after which, in the rectilineal triangle formed by the sides a, b, c, and the angles $A' = A - \frac{1}{3}\varepsilon$, $B' = B - \frac{1}{3}\varepsilon$, $C' = C - \frac{1}{3}\varepsilon$, we shall have the necessary data for determining all its parts. Hence we shall, at the same time, know those of the proposed spherical triangle.

CVII. *Example.* Suppose the angle C were given, and the two sides a and b, namely:

$$C = 123^{0}\ 19'\ 99''.23,$$

$$log\ a = 4.5891503,$$

$$log\ b = 4.5219271.$$

The quantity $\frac{1}{2}\,a\,b\,sin\ C$, which represents the area of the triangle, will have for its logarithm 8.78055; to which adding 2.19612, we shall have $log\ \varepsilon = 0.97667$: hence $\varepsilon = 9''.48$ and $\frac{1}{3}\varepsilon = 3''.16$. This being determined, we are next to solve the rectilineal triangle which has two sides, a and b, given as above, and the included angle $C' = 123^{0}\ 19'\ 96''.07$. We shall follow the method explained in Art. 56.

a	4.5891503	$tang\ (\varphi - 50^{0})$	8.8878392
b	4.5219271	$cot\ \frac{1}{2}C'$	9.8381110
$tang\ \varphi$ . . .	0.0672232	$tang\ \frac{A'-B'}{2}$	8.7259502
$\varphi = 54^{0}\ 90'\ 74''.72$		$\frac{A'-B'}{2} = 3^{0}\ 38'\ 39''.27$	
$\frac{1}{2}C' = 61\ \ 59\ \ 98\ .03$			
$100^{0} - \frac{1}{2}C' = 38\ \ 40\ \ 1\ .97$		$\frac{A'+B'}{2} = 38\ \ 40\ \ 1\ .97$	
		$A' = 41\ \ 78\ \ 41\ .24$	
		$B' = 31\ \ 1\ \ 62\ .70$	

We have yet to determine the third side c; for which there is the equation $c = \frac{a\ sin\ C'}{sin\ A'}$.

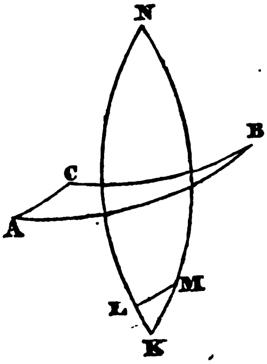
a	4.5891503		
$sin\ A'$	9.7854893		
difference . .	4.8036610	. . . .	4.8036610
$sin\ C'$	9.9705008	$sin\ B'$	9.7182661
$log\ c =$ . . .	4.7741618	$log\ b =$ . . .	4.5219271

Hence in the proposed spherical triangle, the elements sought are,

$$A = 41^\circ\ 78'\ 44''.40$$
$$B = 35\quad 1\quad 65\ .86$$
$$\log c = 4.7741618, \text{ or } c = 59451.256 \text{ metres.}$$

Note. The method explained in this paragraph, may likewise serve for solving triangles which have two sides nearly equal to 200°, and the third side very small. For, producing the long sides NL, NM, (see the next fig.), we shall have a spherical triangle LKM, whose three sides are very small.

§ VI. *Of spherical triangles which have two very acute angles.*



CVIII. Let ABC be the proposed spherical triangle, in which A and B are two very acute angles; let LMN be its polar triangle; so that MN $= 200^\circ - A$ and LN $= 200^\circ - B$. Producing the arcs NM, NL, till they meet in K, it is evident that we shall have KM $=$ A, and KL $=$ B; the triangle LKM will therefore have its sides very small, and may consequently be solved according to the method of the last paragraph. Let A', B', C', be the three angles, and a', b', c', the three sides of the triangle LKM; we shall have

$$A' = MLK = a \qquad a' = KM = A$$
$$B' = LMK = b \qquad b' = LK = B$$
$$C' = LKM = 200^\circ - c \qquad c' = LM = 200^\circ - C.$$

Hence, three known elements in the triangle ABC will give three in the triangle LKM, and, consequently, three likewise in the rectilineal triangle to which the triangle LKM is to be reduced. Now, the latter being solved, we shall have the solution of the triangle LKM; and hence that of the proposed triangle ABC.

CIX. Suppose, for example, that $A = 3^\circ$, $B = 2^\circ$, and the adjacent side $c = 150^\circ$; the data of the triangle LKM, or rather of $A'B'C'$, will be $a' = 3^\circ$, $b' = 2^\circ$, and the included angle $C' = 150^\circ$. By means of these data, we find the spherical excess $\epsilon = \dfrac{\frac{1}{2} a' b' \sin C'}{R}$ $= 333''.21$, and the third of ϵ being taken away from C', the remainder will be $49^\circ\ 98'\ 88''.93$. Hence we are to solve a rectilineal triangle, in which are given the two sides $a' = 30000''$, $b' = 20000''$, and the included angle $C'' = 49^\circ\ 98'\ 88''.93$. We shall find the other two angles $A' = 103^\circ\ 64'\ 86''.33$, $B'' = 46^\circ\ 36'\ 24''.75$, and the third side $c' = 21244''.36$; hence, adding $\frac{1}{3}\epsilon$ to the angle A B of the rectilineal triangle, in order to obtain the angles A', B' of the spherical triangle, we shall have for the required solution

$$\begin{aligned} A' &= a = 103^\circ\ 65'\ 97''.40 \\ B &= b = \ \ 46\ \ \ 37\ \ 35\ .82 \\ C = 200^\circ - c' &= \ 197\ \ \ 87\ \ 55\ .64. \end{aligned}$$

§ VII *Of the regular polygon having seventeen sides.*

CX. We shall terminate these applications of the trigonometrical calculus, by exhibiting (from the excellent work of Gauss, mentioned in the note to Prop. 5. IV.) the method of inscribing a regular polygon of seventeen sides, by merely resolving equations of the second degree.

Put the arc $\frac{200^\circ}{17} = \varphi$; then first we shall have the equation

$$\cos\varphi + \cos 3\varphi + \cos 5\varphi + \cos 7\varphi + \cos 9\varphi + \cos 11\varphi + \cos 13\varphi + \cos 15\varphi = \tfrac{1}{2}.$$

For, calling the first member P, and multiplying all its terms by $2\cos\varphi$, then changing each product of two cosines into cosines of simple arcs by the formula

$$2\cos A \cos B = \cos(A+B) + \cos(A-B),$$

we shall have

$$2P\cos\varphi = 1 + 2\cos 2\varphi + 2\cos 4\varphi + 2\cos 6\varphi + \ldots\ldots + 2\cos 14\varphi + \cos 15\varphi.$$

Now since $17\varphi = 200^\circ$, we have $\cos 2\varphi = \cos(200^\circ - 15\varphi) = -\cos 15\varphi$, $\cos 4\varphi = \cos(200^\circ - 13\varphi) = -\cos 13\varphi$, and so on, to $\cos 16\varphi = -\cos\varphi$. Hence

$$2P\cos\varphi = 1 - 2\cos 15\varphi - 2\cos 13\varphi - 2\cos 11\varphi \ldots - 2\cos 3\varphi - \cos\varphi,$$

or $2P\cos\varphi = 1 + \cos\varphi - 2P$, or $2P(1 + \cos\varphi) = 1 + \cos\varphi$. Hence $P = \frac{1}{2}$.

This being settled, we divide the sum of the terms composing P into two parts, namely,

$$\begin{aligned} x &= \cos 3\varphi + \cos 5\varphi + \cos 7\varphi + \cos 11\varphi \\ y &= \cos\varphi + \cos 9\varphi + \cos 13\varphi + \cos 15\varphi. \end{aligned}$$

We shall then have, in the first place, $x + y = \frac{1}{2}$; next we multiply the four terms of x by the four terms of y; and changing the products of cosines into cosines of simple arcs, we obtain, after all those reductions,

$$\begin{aligned} xy &= \ \ 2(\cos 2\varphi + \cos 4\varphi + \cos 6\varphi \ldots + \cos 16\varphi) \\ \text{or } xy &= -2(\cos 15\varphi + \cos 13\varphi + \cos 11\varphi \ldots + \cos\varphi) \end{aligned}$$

or, lastly, $xy = -1$.

By means of these equations, we find

$$x = \tfrac{1}{4} + \tfrac{1}{4}\sqrt{17}; \quad y = \tfrac{1}{4} - \tfrac{1}{4}\sqrt{17}.$$

Now, if we again divide the sums x and y, each into two parts, namely,

$$\begin{array}{ll} x = s + t & y = u + z \\ s = \cos 3\varphi + \cos 5\varphi & u = \cos\varphi + \cos 13\varphi \\ t = \cos 7\varphi + \cos 11\varphi & z = \cos 9\varphi + \cos 15\varphi, \end{array}$$

we shall, in like manner, find

$$s\,t = -\tfrac{1}{4} \qquad u\,z = -\tfrac{1}{4}.$$

So that we shall be able to determine the four numbers, s, t, u, z, by means of two equations of the second degree.

In fine, knowing $\cos\varphi + \cos 13\,\varphi = u$ and $\cos\varphi \cos 13\,\varphi = \frac{1}{2}(\cos 12\,\varphi + \cos 14\,\varphi) = -\frac{1}{2}(\cos 3\,\varphi + \cos 5\,\varphi) = -\frac{1}{2}s$, by a fourth equation of the second degree, we shall obtain the value of $\cos\varphi$, and from it the value of a side of the proposed polygon, which value is $2\sin\varphi$, or $2\sqrt{(1 - \cos^2\varphi)}$.

As for the method which has guided us in separating these various equations, it belongs to a very delicate theory, founded upon the indeterminate analysis, and for the developement of which we refer to the work of Gauss, or to the *Essai sur la Théorie des Nombres*, second edition. The reader will there find the complete demonstration of this very beautiful and very general theorem:

“ If the number n is prime, and if $n - 1$ result from the product of
“ the prime factors $2^{\alpha}\,3^{\beta}\,5^{\gamma}$ &c., the division of the circle into n equal
“ parts may always be reduced to the solution of α equations of the
“ second degree, β of the third, γ of the fifth, and so on.”

TRIGONOMETRICAL TABLES.

The three following Tables exhibit, in one view, a summary of the most important doctrines of Trigonometry, and may be useful for enabling the student to apply them more readily to practice. The formulas are extracted from Cagnoli's *Trigonométrie :* they have either already been demonstrated in the foregoing treatise, or may easily be deduced from some part of it.—Ed

TABLE I.

The most important Analytical Values of Trigonometrical Lines.

Values of Sin A.

1. Formula . . . $\cos A \tan A$
2. $\dfrac{\cos A}{\cot A}$
3. $\sqrt{(1-\cos^2 A)}$
4. $\dfrac{1}{\sqrt{(1+\cot^2 A)}}$
5. $\dfrac{\tan A}{\sqrt{(1+\tan^2 A)}}$
6. $2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A$
7. $\sqrt{\dfrac{1-\cos 2A}{2}}$
8. $\dfrac{2 \tan \frac{1}{2}A}{1+\tan^2 \frac{1}{2}A}$
9. $\dfrac{2}{\cot \frac{1}{2}A + \tan \frac{1}{2}A}$
10. $\dfrac{\sin(30^\circ + A) - \sin(30^\circ - A)}{\sqrt{3}}$
11. $2 \sin^2(45^\circ + \tfrac{1}{2}A) - 1$
12. $1 - 2 \sin^2 (45^\circ + \tfrac{1}{2}A)$
13. $\dfrac{1 - \tan^2(45^\circ - \frac{1}{2}A}{1 + \tan^2(45^\circ - \frac{1}{2}A)}$
14. $\dfrac{\tan(45^\circ + \frac{1}{2}A) - \tan(45^\circ - \frac{1}{2}A)}{\tan(45^\circ + \frac{1}{2}A) + \tan(45^\circ - \frac{1}{2}A)}$
15. $\sin(60 + A) - \sin(60^\circ - A)$

Values of Cos A.

16. $\frac{\sin A}{\text{tang } A}$

17. $\sin A \cot A$

18. $\sqrt{(1-\sin^2 A)}$

19. $\frac{1}{\sqrt{(1+\text{tang}^2 A)}}$

20. $\frac{\cot A}{\sqrt{(1+\cot^2 A)}}$

21. $\cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A$

22. $1 - 2\sin^2 \frac{1}{2}A$

23. $2\cos^2 \frac{1}{2}A - 1$

24. $\sqrt{\frac{1+\cos 2A}{2}}$

25. $\frac{1-\text{tang}^2 \frac{1}{2}A}{1+\text{tang}^2 \frac{1}{2}A}$

26. $\frac{\cot \frac{1}{2}A - \text{tang } \frac{1}{2}A}{\cot \frac{1}{2}A + \text{tang } \frac{1}{2}A}$

27. $\frac{1}{1+\text{tang } A \text{ tang } \frac{1}{2}A}$

28. $\frac{2}{\text{tang } (45^\circ+\frac{1}{2}A)+\cot (45^\circ+\frac{1}{2}A)}$

29. $2\cos (45^\circ+\frac{1}{2}A) \cos (45^\circ-\frac{1}{2}A)$

30. $\cos (60^\circ+A)+\cos (60^\circ-A)$

Values of Tang A.

31. $\frac{\sin A}{\cos A}$

32. $\frac{1}{\cot A}$

33. $\sqrt{\left(\frac{1}{\cos^2 A}-1\right)}$

34. $\frac{\sin A}{\sqrt{(1-\sin^2 A)}}$

35. $\frac{\sqrt{(1-\cos^2 A)}}{\cos A}$

36. $\frac{2 \text{ tang } \frac{1}{2}A}{1-\text{tang}^2 \frac{1}{2}A}$

37. $\frac{2 \cot \frac{1}{2}A}{\cot^2 \frac{1}{2}A - 1}$

38. $\frac{2}{\cot \frac{1}{2}A - \text{tang } \frac{1}{2}A}$

39. $\cot A - 2 \cot 2A$

40. $\frac{1-\cos 2A}{\sin 2A}$

41. $\frac{\sin 2A}{1+\cos 2A}$

42. $\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$

43. $\frac{\text{tang } (45^\circ+\frac{1}{2}A) - \text{tang } (45^\circ-\frac{1}{2}A)}{2}$

TABLE II.

Analytical Formulas for solving, in all Cases, a Rectilineal Triangle ABC, *of which three Parts are given.*

Values of AB.	*Values of* AC.	*Values of* BC.
1. FORMULA . . . $\frac{BC \sin C}{\sin A}$	10. $\frac{AB \sin B}{\sin C}$	19. $\frac{AC \sin A}{\sin B}$
2. $\frac{AC \sin C}{\sin B}$	11. $\frac{BC \sin B}{\sin A}$	20. $\frac{AB \sin A}{\sin C}$
3. $\frac{BC}{\cos B + \sin B \cot C}$	12. $\frac{AB}{\cos A + \sin A \cot B}$	21. $\frac{AC}{\cos C + \sin C \cot A}$
4. $\frac{AC}{\cos A + \sin A \cot C}$	13. $\frac{BC}{\cos C + \sin C \cot B}$	22. $\frac{AB}{\cos B + \sin B \cot A}$
5. $BC \cos B + BC \sin B \cot A$	14. $AB \cos A + AB \sin A \cot C$	23. $AC \cos C + AC \sin C \cot B$
6. $AC \cos A + AC \sin A \cot B$	15. $BC \cos C + BC \sin C \cot A$	24. $AB \cos B + AB \sin B \cot C$
7. $\surd(BC^2 + AC^2 - 2\,BC \times AC \cos C)$	16. $\surd(BC^2 + AB^2 - 2\,BC \times AB \cos B)$	25. $\surd(AB^2 + AC^2 - 2\,AB \times AC \cos A)$
8. $BC \cos B \pm \surd(AC^2 - BC^2 \sin^2 B)$	17. $AB \cos A \pm \surd(BC^2 - AB^2 \sin^2 A)$	26. $AC \cos C \pm \surd(AB^2 - AC^2 \sin^2 C)$
9. $AC \cos A \pm \surd(BC^2 - AC^2 \sin^2 A)$	18. $BC \cos C \pm \surd(AB^2 - BC^2 \sin^2 C)$	27. $AB \cos B \pm \surd(AC^2 - AB^2 \sin^2 B)$

Values of sin A.	*Values of* cos A.	*Values of* tang A.
28. $\frac{BC \sin C}{AB}$	37. $\frac{\pm\sqrt{(AB^2 - BC^2 \sin^2 C)}}{AB}$	46. $\frac{BC \sin C}{\mp\sqrt{(AB^2 - BC^2 \sin^2 C)}}$
29. $\frac{BC \sin B}{AC}$	38. $\frac{\pm\sqrt{(AC^2 - BC^2 \sin^2 B)}}{AC}$	47. $\frac{BC \sin B}{\mp\sqrt{(AC^2 - BC^2 \sin^2 B)}}$
30. $\sin(B+C)$	39. $-\cos(B+C)$	48. $-\text{tang}\,(B+C)$
31. $\sin B \cos C + \cos B \sin C$	40. $\sin B \sin C - \cos B \cos C$	49. $\frac{\text{tang}\,B + \text{tang}\,C}{\text{tang}\,B\,\text{tang}\,C - 1}$
32. $\frac{BC \sin C}{\sqrt{(BC^2 + AC^2 - 2\,BC \times AC \cos C)}}$	41. $\frac{AC - BC \cos C}{\sqrt{(BC^2 + AC^2 - 2\,BC \times AC \cos C)}}$	50. $\frac{BC \sin C}{AC - BC \cos C}$
33. $\frac{BC \sin B}{\sqrt{(BC^2 + AB^2 - 2\,BC \times AB \cos B)}}$	42. $\frac{AB - BC \cos B}{\sqrt{(BC^2 + AB^2 - 2\,BC \times AB \cos B)}}$	51. $\frac{BC \sin B}{AB - BC \cos B}$
34. $\sqrt{1 - \left(\frac{AB^2 + AC^2 - BC^2}{2\,AB \times AC}\right)^2}$	43. $\frac{AB^2 + AC^2 - BC^2}{2\,AB \times AC}$	52. $\pm\sqrt{\left(\frac{2\,AB \times AC}{AB^2 + AC^2 - BC^2}\right)^2 - 1}$
35. $\frac{\sin C\,(AC \cos C \pm \sqrt{AB^2 - AC^2 \sin^2 C})}{AB}$	44. $\frac{AC \sin^2 C \mp \cos C \sqrt{(AB^2 - AC^2 \sin^2 C)}}{AB}$	53. $\frac{AC \cos C \pm \sqrt{(AB^2 - AC^2 \sin^2 C)}}{AC \sin C \mp \cot C \sqrt{(AB^2 - AC^2 \sin^2 C)}}$
36. $\frac{\sin B\,(AB \cos B \pm \sqrt{AC^2 - AB^2 \sin^2 B})}{AC}$	45. $\frac{AB \sin^2 B \mp \cos B \sqrt{(AC^2 - AB^2 \sin^2 B)}}{AC}$	54. $\frac{AB \cos B \pm \sqrt{(AC^2 - AB^2 \sin^2 B)}}{AB \sin B \pm \cot B \sqrt{(AC^2 - AB^2 \sin^2 B)}}$

Continuation of Table II.

Values of sin B	*Values of* cos B	*Values of* tang B
55. **Formula** . . . $\frac{AC \sin A}{BC}$	64. $\frac{\pm\sqrt{(BC^2 - AC^2 \sin^2 A)}}{BC}$	73. $\frac{AC \sin A}{\pm\sqrt{(BC^2 - AC^2 \sin^2 A)}}$
56. $\frac{AC \sin C}{AB}$	65. $\frac{\pm\sqrt{(AB^2 - AC^2 \sin^2 C)}}{AB}$	74. $\frac{AC \sin C}{\pm\sqrt{(AB^2 - AC^2 \sin^2 C)}}$
57. $\sin (A+C)$	66. $-\cos (A+C)$	75. $-\text{tang}\,(A+C)$
58. $\sin A \cos C + \cos A \sin C$	67. $\sin A \sin C - \cos A \cos C$	76. $\frac{\text{tang}\,A + \text{tang}\,C}{\text{tang}\,A\ \text{tang}\,C - 1}$
59. $\frac{AC \sin A}{\sqrt{(AB^2 + AC^2 - 2\,AB + AC \cos A)}}$	68. $\frac{AB - AC \cos A}{\sqrt{AB^2 + AC^2 - 2\,AB \times AC \cos A)}}$	77. $\frac{AC \sin A}{AB - AC \cos A}$
60. $\frac{AC \sin C}{\sqrt{(BC^2 + AC^2 - 2\,BC + AC \cos C)}}$	69. $\frac{BC - AC \cos C}{\sqrt{(BC^2 + AC^2 - 2\,BC \times AC \cos C)}}$	78. $\frac{AC \sin C}{BC - AC \cos C}$
61. $\sqrt{1 - \left(\frac{BC^2 + AB^2 - AC^2}{2\,BC \times AB}\right)^2}$	70. $\frac{BC^2 + AB^2 - AC^2}{2\,BC \times AB}$	79. $\pm\sqrt{\left(\frac{2\,BC \times AB}{BC^2 + AB^2 - AC^2}\right)^2 - 1}$
62. $\frac{\sin A\,(AB \cos A \pm \sqrt{BC^2 - AB^2 \sin^2 A})}{BC}$	71. $\frac{AB \sin^2 A \mp \cos C \sqrt{(BC^2 - AB^2 \sin^2 A)}}{BC}$	80. $\frac{AB \cos A \pm \sqrt{(BC^2 - AB^2 \sin^2 A)}}{AB \sin A \mp \cot A \sqrt{(BC^2 - AB^2 \sin^2 A)}}$
63. $\frac{\sin C\,(BC \cos C \pm \sqrt{AB^2 - BC^2 \sin^2 C})}{AB}$	72. $\frac{BC \sin^2 C \mp \cos C \sqrt{(AB^2 - BC^2 \sin^2 C)}}{AB}$	81. $\frac{BC \cos C \pm \sqrt{(AB^2 - BC^2 \sin^2 C)}}{BC \sin C \mp \cot C \sqrt{(AB^2 - BC^2 \sin^2 C)}}$

Values of sin C	*Values of* cos C	*Values of* tang C
82. $\frac{AB \sin B}{AC}$	91. $\frac{\pm\sqrt{(AC^2 - AB^2 \sin^2 B)}}{AC}$	100. $\frac{AB \sin B}{\pm\sqrt{(AC^2 - AB^2 \sin^2 B)}}$
83. $\frac{AB \sin A}{BC}$	92. $\frac{\pm\sqrt{(BC^2 - AB^2 \sin^2 A)}}{BC}$	101. $\frac{AB \sin A}{\pm\sqrt{(BC^2 - AB^2 \sin^2 A)}}$
84. $\sin (A+B)$	93. $-\cos (A+B)$	102. $-\operatorname{tang} (A+B)$
85. $\sin A \cos B + \cos A \sin B$	94. $\sin A \sin B - \cos A \cos B$	103. $\frac{\operatorname{tang} A + \operatorname{tang} B}{\operatorname{tang} A \operatorname{tang} B - 1}$
86. $\frac{AB \sin B}{\sqrt{(BC^2 + AB^2 - 2\, BC \times AB \cos B)}}$	95. $\frac{BC - AB \cos B}{\sqrt{(BC^2 + AB^2 - 2\, BC \times AB \cos B)}}$	104. $\frac{AB \sin B}{BC - AB \cos B}$
87. $\frac{AB \sin A}{\sqrt{(AB^2 + AC - 2\, AB \times AC \cos A)}}$	96. $\frac{AC - AB \cos A}{\sqrt{(AB^2 + AC^2 - 2\, AB \times AC \cos A)}}$	105. $\frac{AB \sin A}{AC - AB \cos A}$
88. $\sqrt{1 - \left(\frac{BC^2 + AC^2 - AB^2}{2\, BC \times AC}\right)^2}$	97. $\frac{BC^2 + AC^2 - AB^2}{2\, BC \times AC}$	106. $\pm \sqrt{\left(\frac{2\, BC \times AC}{BC^2 + AC^2 - AB^2}\right)^2 - 1}$
89. $\frac{\sin B (BC \cos B \pm \sqrt{AC^2 - BC^2 \sin^2 B})}{AC}$	98. $\frac{BC \sin^2 B \mp \cos B \sqrt{(AC^2 - BC^2 - \sin^2 B)}}{AC}$	107. $\frac{BC \cos B \pm \sqrt{(AC^2 - BC^2 \sin^2 B)}}{BC \sin B + \cot B \sqrt{(AC^2 - BC^2 \sin^2 B)}}$
90. $\frac{\sin A (AC \cos A + \sqrt{BC^2 - AC^2 \sin^2 A})}{BC}$	99 $\frac{AC \sin^2 A \mp \cos A \sqrt{(BC^2 - AC^2 \sin^2 A)}}{BC}$	108. $\frac{AC \cos A \pm \sqrt{(BC^2 - AC^2 \sin^2 A)}}{AC \sin A \mp \cot A \sqrt{(BC^2 - AC^2 \sin^2 A)}}$

TABLE III.

Analytical Expressions for the Trigonometrical Lines of a Spherical Triangle ABC.

1. Formula . . . . $\sin A = \frac{\sin BC \sin C}{\sin AB}$

2. $\sin A = \frac{\sin BC \sin B}{\sin AC}$

3. $\sin B = \frac{\sin AC \sin A}{\sin BC}$

4. $\sin B = \frac{\sin AC \sin C}{\sin AB}$

5. $\sin C = \frac{\sin AB \sin B}{\sin AC}$

6. $\sin C = \frac{\sin AB \sin A}{\sin BC}$

7. $\cos A = \frac{\cos BC - \cos AB \cos AC}{\sin AB \sin AC}$

8. $\cos A = \cos BC \sin B \sin C - \cos B \cos C$

19. $\sin BC = \frac{\sin AB \sin A}{\sin C}$

20. $\sin BC = \frac{\sin AC \sin A}{\sin B}$

21. $\sin AC = \frac{\sin BC \sin B}{\sin A}$

22. $\sin AC = \frac{\sin AB \sin B}{\sin C}$

23. $\sin AB = \frac{\sin AC \sin C}{\sin B}$

24. $\sin AB = \frac{\sin BC \sin C}{\sin A}$

25. $\cos BC = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$

26. $\cos BC = \cos A \sin AB \sin AC + \cos AB \cos AC$

9. $$\cos B = \frac{\cos AC - \cos BC \cos AB}{\sin BC \sin AB}$$

10. $$\cos B = \cos AC \sin A \sin C - \cos A \cos C$$

11. $$\cos C = \frac{\cos AB - \cos BC \cos AC}{\sin BC \sin AC}$$

12. $$\cos C = \cos AB \sin A \sin B - \cos A \cos B$$

13. $$\mathrm{tang}\, A = \frac{\sin B}{\sin AB \cot BC - \cos AB \cos B}$$

14. $$\mathrm{tang}\, A = \frac{\sin C}{\sin AC \cot BC - \cos AC \cos C}$$

15. $$\mathrm{tang}\, B = \frac{\sin C}{\sin BC \cot AC - \cos BC \cos C}$$

16. $$\mathrm{tang}\, B = \frac{\sin A}{\sin AB \cot AC - \cos AB \cos A}$$

17. $$\mathrm{tang}\, C = \frac{\sin A}{\sin AC \cot AB - \cos AC \cos A}$$

18. $$\mathrm{tang}\, C = \frac{\sin B}{\sin BC \cot AB - \cos BC \cos B}$$

27. $$\cos AC = \frac{\cos B + \cos A \cos C}{\sin A \sin C}$$

28. $$\cos AC = \cos B \sin BC \sin AB + \cos BC \cos AB.$$

29. $$\cos AB = \frac{\cos C + \cos A \cos B}{\sin A \sin B}$$

30. $$\cos AB = \cos C \sin BC \sin AC + \cos BC \cos AC$$

31. $$\mathrm{tang}\, BC = \frac{\sin AB}{\sin B \cot A + \cos B \cos AB}$$

32. $$\mathrm{tang}\, BC = \frac{\sin AC}{\sin C \cot A + \cos C \cos AC}$$

33. $$\mathrm{tang}\, AC = \frac{\sin BC}{\sin C \cot B + \cos C \cos BC}$$

34. $$\mathrm{tang}\, AC = \frac{\sin AB}{\sin A \cot B + \cos A \cos AB}$$

35. $$\mathrm{tang}\, AB = \frac{\sin AC}{\sin A \cot C + \cos A \cos AC}$$

36. $$\mathrm{tang}\, AB = \frac{\sin BC}{\sin B \cot C + \cos B \cos BC}$$

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